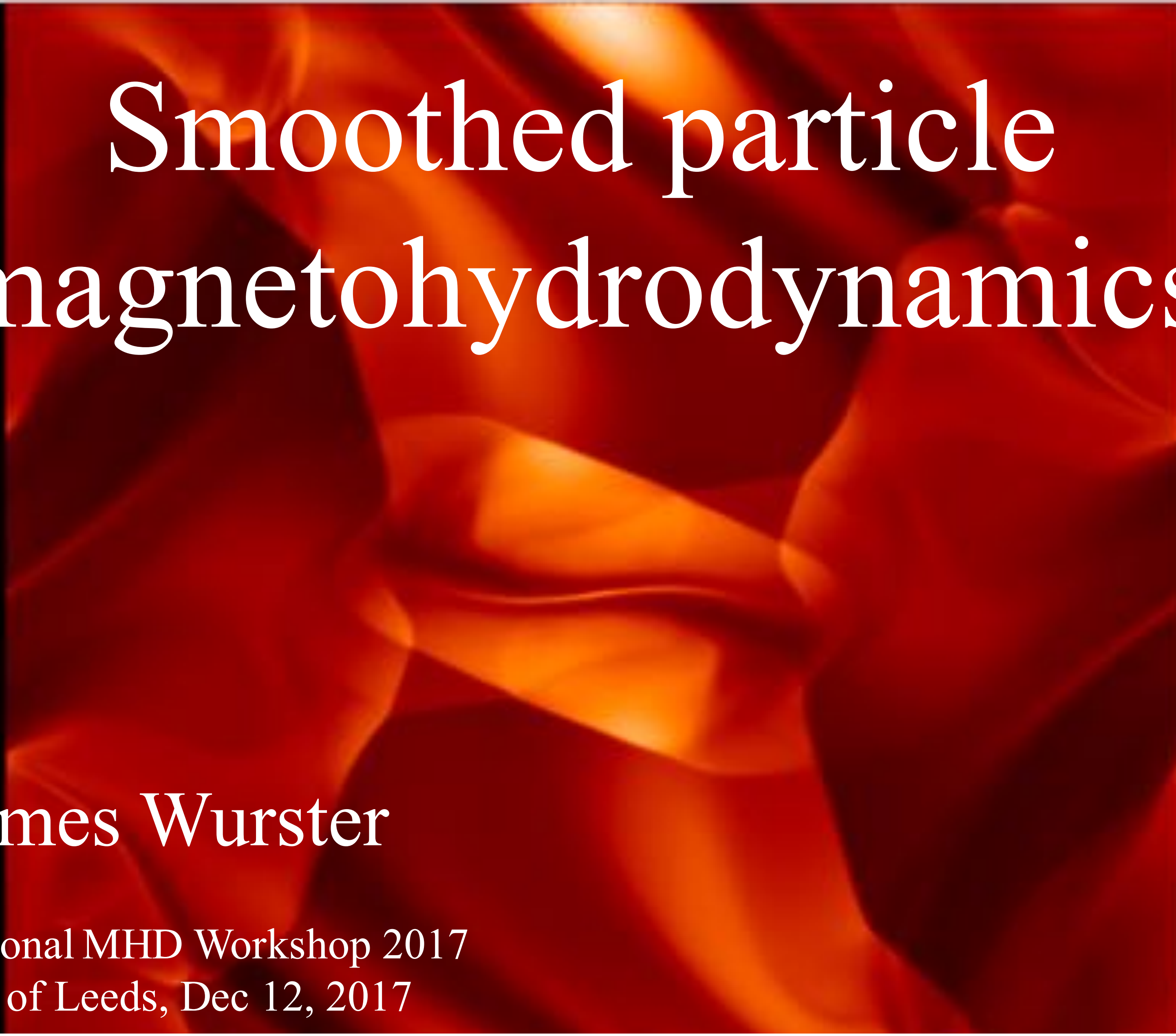
# Smoothed particle magnetohydrodynamics



Dr. James Wurster

Computational MHD Workshop 2017
University of Leeds, Dec 12, 2017

UNIVERSITY OF EXETER erc DiRAC PHANTOMSPH

# SPH: Historical Overview



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## A numerical approach to the testing of the fission hypothesis

L. B. Lucy[a]
*Institute of Astronomy, Cambridge, United Kingdom*
*European Southern Observatory, Geneva, Switzerland*
(Received 12 August 1977; revised 16 September 1977)

**A finite-size particle scheme for the numerical solution of two- and three-dimensional gas dynamical problems of astronomical interest is described and tested. The scheme is then applied to the fission problem for optically thick protostars. Results are given, showing the evolution of one such protostar from an initial state as a single, rotating star to a final state as a triple system whose components contain 60% of the original mass. The decisiveness of this numerical test of the fission hypothesis and its relevance to observed binaries are briefly discussed.**

### INTRODUCTION

THE hypothesis that fission is the mechanism by which close binaries are formed has regained favor in recent years. Those responsible for this revival (Lynden-Bell 1964, 1965; James 1964; Stoeckly 1965; Roxburgh 1966; Bodenheimer and Ostriker 1970; Lebovitz 1972, 1974) have rebutted earlier theoretical objections (see also Ostriker 1970) and have discussed the hypothesis in the context of our current understanding of pre-main-sequence evolution. The early history of the fission hypothesis and the related investigations into the figures of equilibrium of rotating liquids has been summarized by Chandrasekhar (1969).

Although fission is now commonly considered to be the most likely explanation for the existence of close binaries, the hypothesis cannot be regarded as proved until the evolution of a rotating protostar has been followed from an initial state as a single star to a final state as a detached binary system. This is a formidable problem, however, since it requires the ability to compute the three-dimensional motion of a self-gravitating, compressible gas. Fortunately, some simplifying circumstances make it less than forbidding. First, the high frequency of close binaries over a wide mass range surely implies that no special characteristics of the properties of stellar matter are essential to binary formation; consequently, these properties need not be treated accurately.

A second and crucial simplification concerns the spatial resolution of the calculation. Because the initial departure from axial symmetry is due to the onset of dynamical overstability for a mode of low order, we might reasonably hope that the subsequent evolution can be adequately followed with a low-resolution description of the protostar's structure. If this is indeed so, the problem can be tackled with present-day computers.

On the assumption, therefore, that a decisive test of the fission hypothesis might be provided by a three-dimensional gas dynamical calculation of low spatial resolution, the bulk of this paper is devoted to describing (Sec. II) and testing (Sec. III) a numerical scheme for carrying out such calculations. This scheme is then used (Sec. IV) to follow the contraction of a rotating protostar and results illustrating the fission mechanism are obtained.

### I. ASSUMPTIONS AND EQUATIONS

In this section, after stating our assumptions, we derive the basic equations in the form used when applying the numerical technique of Sec. II.

(*a*) *Assumptions.* A rotating, axisymmetric, optically thick protostar of homogeneous composition will be the starting point of the calculation, and this protostar's evolution will be followed up to and beyond the point of instability to a nonaxisymmetric perturbation. To ensure that contraction does not halt prior to this point, energy generation by nuclear burning will be omitted. Accordingly, the basic equations are those describing the motion of a self-gravitating, compressible gas with entropy changes occurring only as a result of radiative conduction.

In accordance with the argument that the detailed properties of stellar matter cannot be of decisive importance, we assume that the matter is a fully ionized perfect gas and that radiation pressure may be neglected; the ratio of specific heats $\gamma$ and the mean molecular weight $\mu$ are then constants. In addition, we assume that the opacity $\kappa$ is independent of state variables.

(*b*) *Units.* In the interest of computational accuracy, it is useful to express dimensions in terms of a time-dependent length scale $R(t)$ chosen so as to largely eliminate the protostar's contraction. We also adopt $\mathcal{M}$, the protostar's mass, as the unit of mass, $\tau_* = (R^3/G\mathcal{M})^{1/2}$ as the unit of time, and $T_* = (\mu m_H/k)(G\mathcal{M}/R)$ as the unit of temperature. In terms of these basic units, we now take $R/\tau_*$ to be the unit of velocity, $1/\tau_*$ to be

[a] Permanent address: Department of Astronomy, Columbia University, New York, NY 10027.





*Mon. Not. R. astr. Soc.* (1977) **181**, 375–389

## Smoothed particle hydrodynamics: theory and application to non-spherical stars

R. A. Gingold and J. J. Monaghan* *Institute of Astronomy, Madingley Road, Cambridge, CB3 0HA*

Received 1977 May 5, in original form February 2

**Summary.** A new hydrodynamic code applicable to a space of an arbitrary number of dimensions is discussed and applied to a variety of polytropic stellar models. The principal feature of the method is the use of statistical techniques to recover analytical expressions for the physical variables from a known distribution of fluid elements. The equations of motion take the form of Newtonian equations for particles. Starting with a non-axisymmetric distribution of approximately 80 particles in three dimensions, the method is found to reproduce the structure of uniformly rotating and magnetic polytropes to within a few per cent. The method may be easily extended to deal with more complicated physical models.

### 1 Introduction

Many of the most interesting problems in astrophysics involve systems with large departures from spherical symmetry. This may occur either because the initial state lacks spherical symmetry, as in the case of a protostar forming from a dense interstellar cloud, or because non-spherical forces arising from rotation or magnetic fields, as in the case of the fission of a rotating star, play an important part in the dynamics. Frequently these sources of non-spherical symmetry will be found combined.

Because of the complexity of these systems numerical methods are required to follow their evolution. However, the standard finite difference representations of the continuum equations are of limited use, because of the very large number of grid points required to treat each coordinate on an equal footing. If, for example, 20 points along the radial direction give adequate accuracy for a spherical polytrope, we may require $(20)^3$ such points to give the same accuracy for a highly distorted polytrope. This difficulty is mirrored in the evaluation of multiple integrals.

For the astrophysical problems a numerical method which allows reasonable accuracy for a small number of points is required. Ideally it should also be simple to program and robust. An early attempt to provide such an alternative to the standard finite difference method was made by Pasta & Ulam (1959). They replaced the continuous fluid by a fictitious set of

* Permanent address: Mathematics Department, Monash University, Clayton, Victoria 3168, Australia.

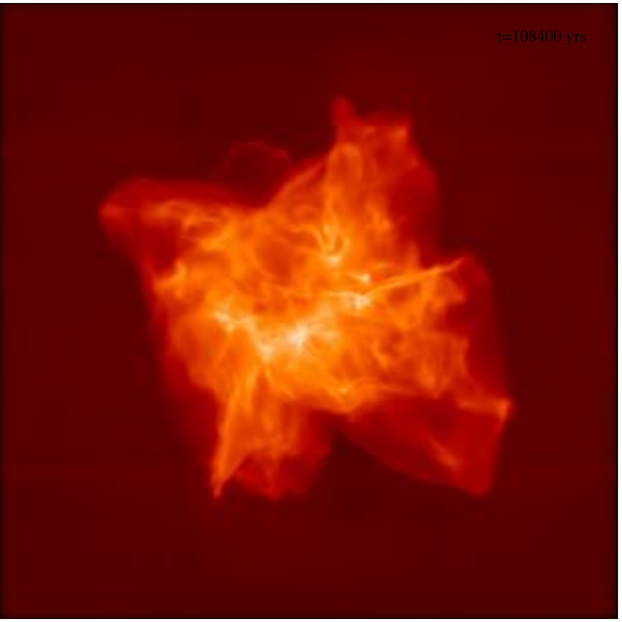
# *SPH: Historical Overview*

- Additions to SPH throughout time
  - Fluid dynamics
  - Gravity (in the original version, but not always included)
  - Radiation
  - Magnetic fields
  - Multi-fluid physics
  - One-fluid physics
  - Pressure-less particles

- Primary reference: D. Price. Feb 2012.
  Smoothed particle hydrodynamics and magnetohydrodynamics . J Comp Phys. 759, 231.

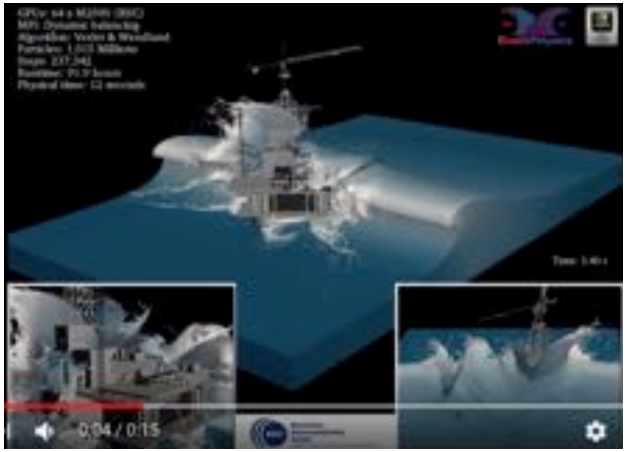
# *SPH: Applications*

Astrophysics




sphNG

Engineering



dualSPHyiscs

https://www.youtube.com/watch?v=B8mP9E75D08

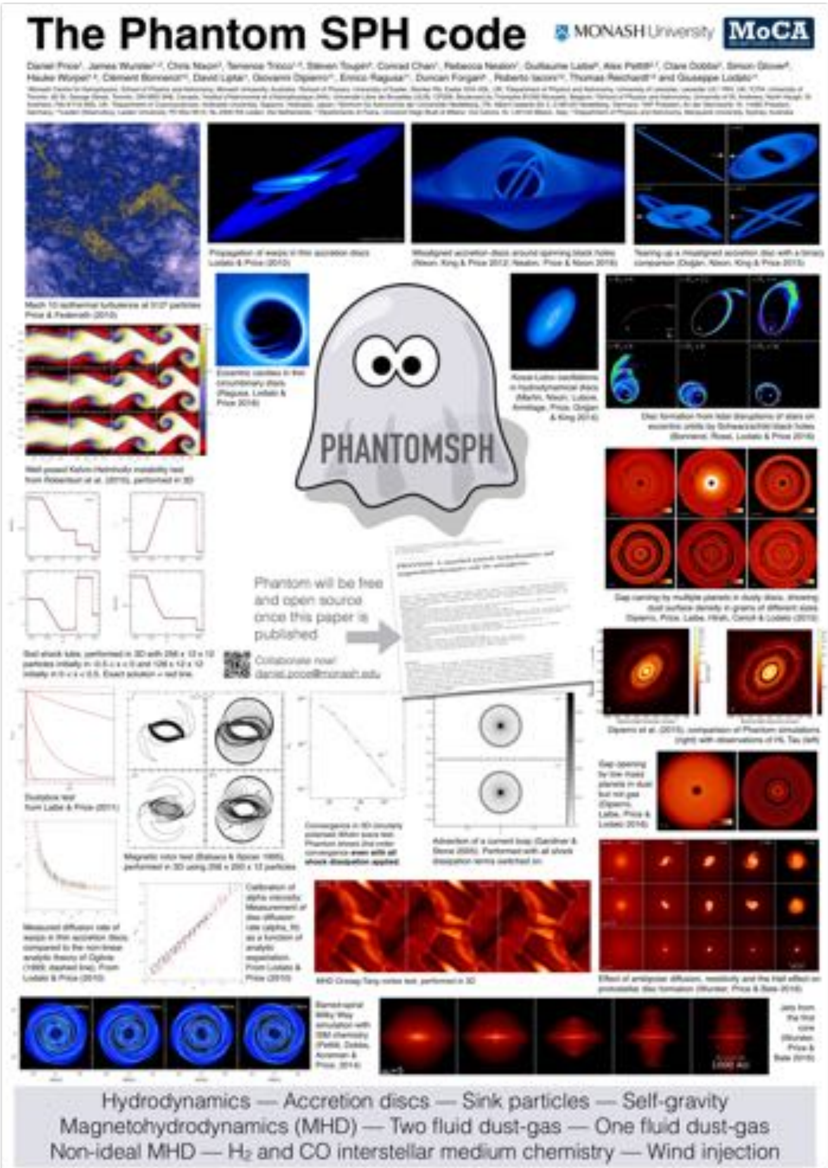
Gaming/Movies

(unknown)

# *SPH: SPHERIC: Annual meeting*

- SPHERIC: European Research Community of all things SPH (mostly engineering, some astrophysics)

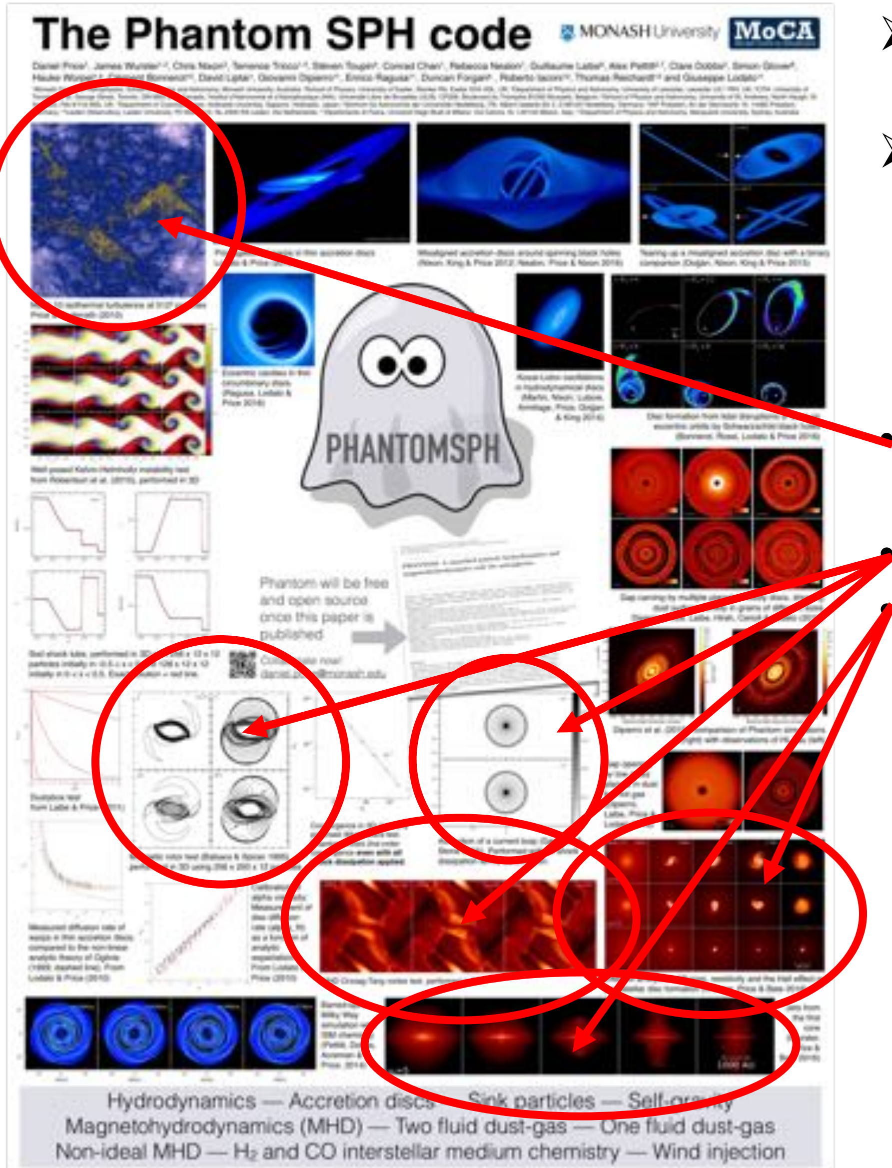
# *Phantom*




- Publically available at https://phantomsph.bitbucket.io
- Reference:
  D. J. Price, **J. Wurster**, C. Nixon,
  T. S. Tricco, and 22 others.
  (arXiv:1702.03930)

- Contains only the ``best” algorithms
  - e.g. one integration scheme, one artificial viscosity algorithm, etc…
- Algorithms can be turned off/on as required, and are fully parameterisable

# *Phantom*

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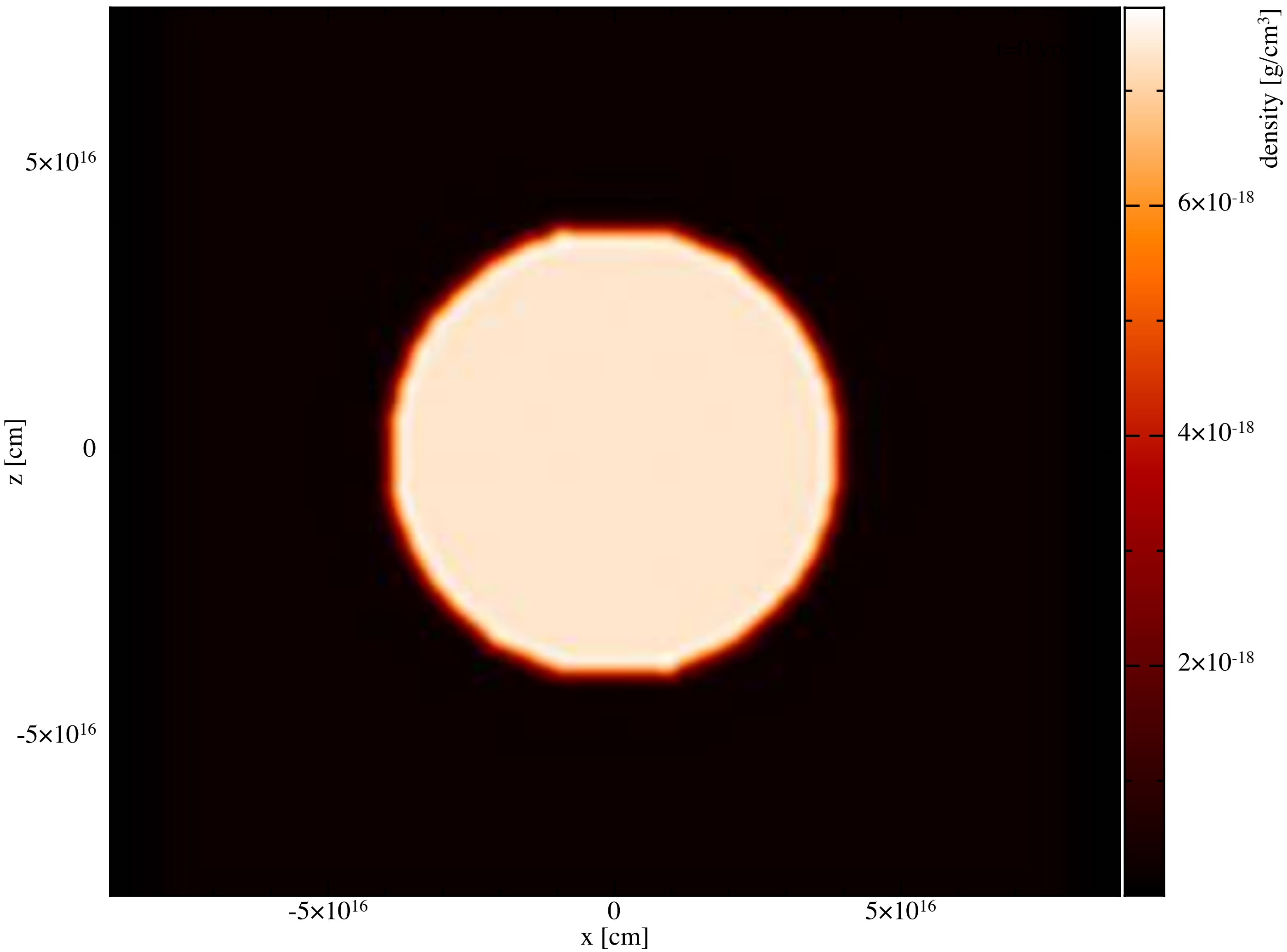
- Turbulence
  (e.g. Tricco, Price & Federrath 2016)
- Test problems
- Star formation (including non-ideal MHD)
  (e.g. Wurster, Price & Bate 2016, 2017)

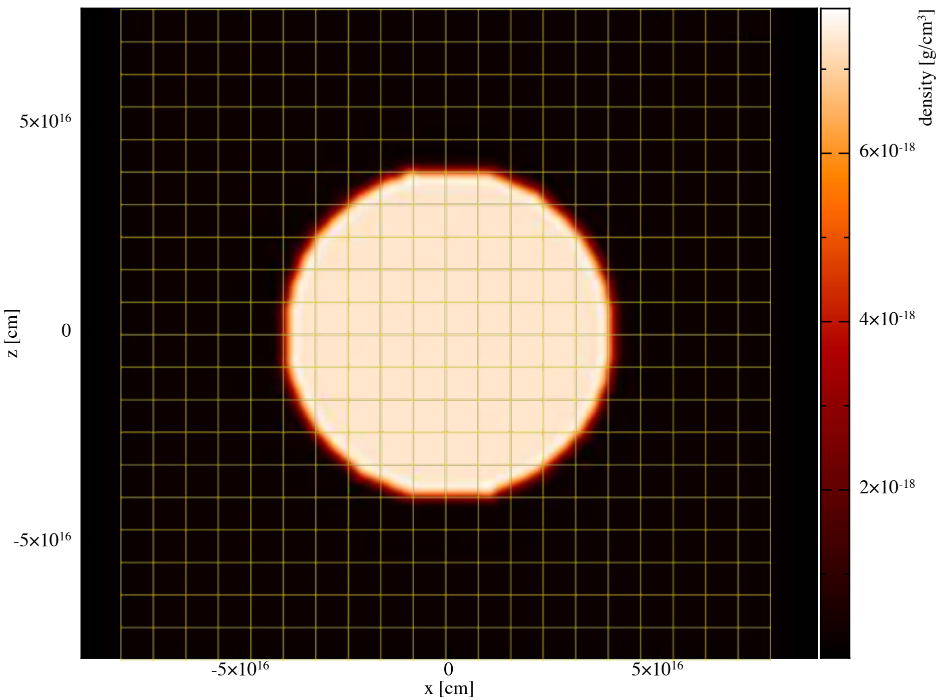
# *SPH vs Grid: Dividing the domain*

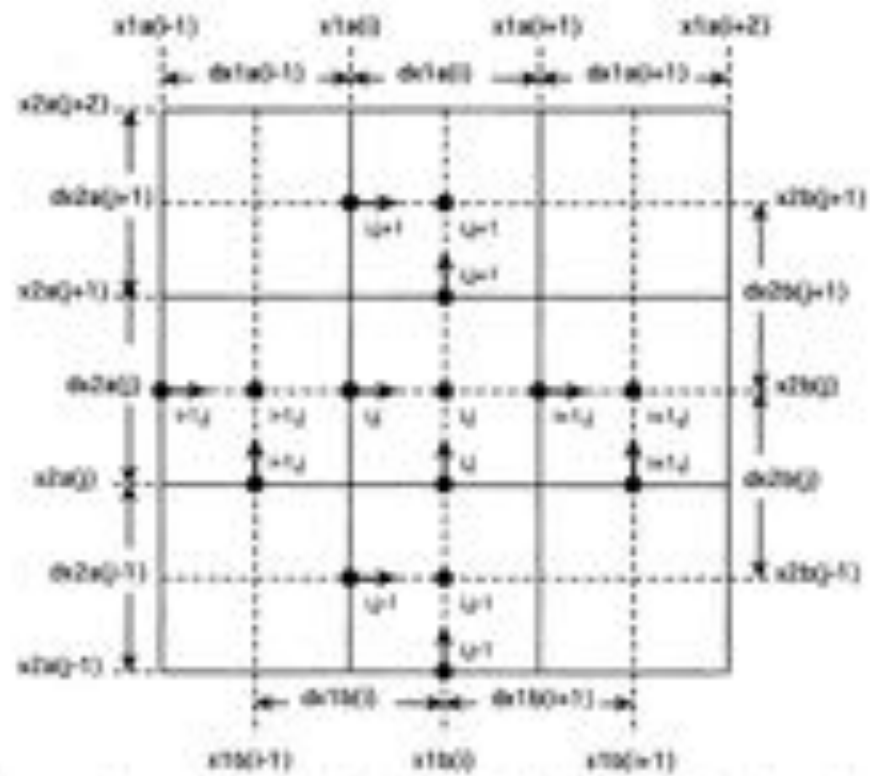
- Given a domain, how do we divide it up?



# *SPH vs Grid: Dividing the domain: Grid*

- Where are the characteristics calculated?
  - Eulerian Grid
    - Cells of well-defined position and volume
    - Evolve scalars at cell-centres
    - Evolve Vectors at cell interfaces



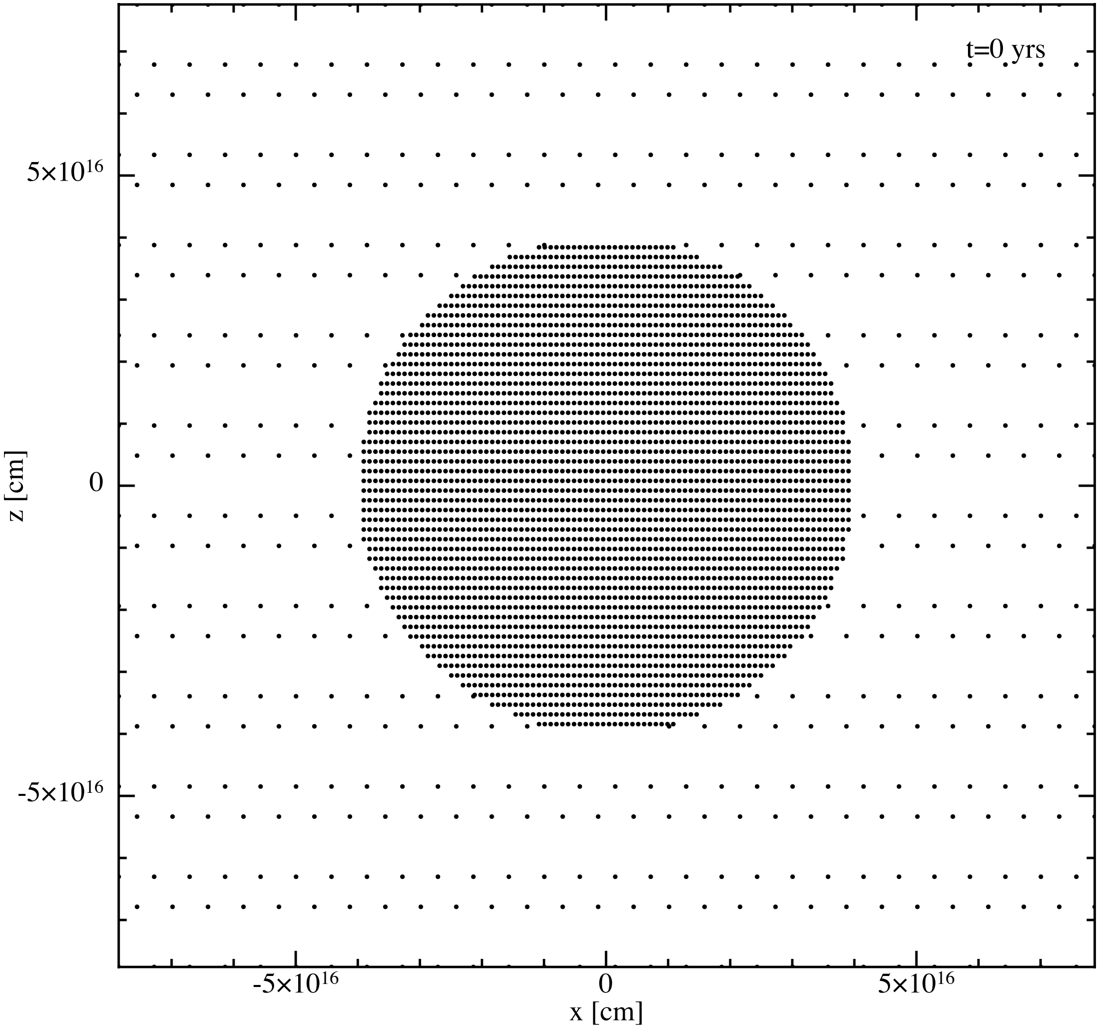





Grid from Zeus2D (Stone & Norman, 1992)

# *SPH vs Grid: Dividing the domain: SPH*

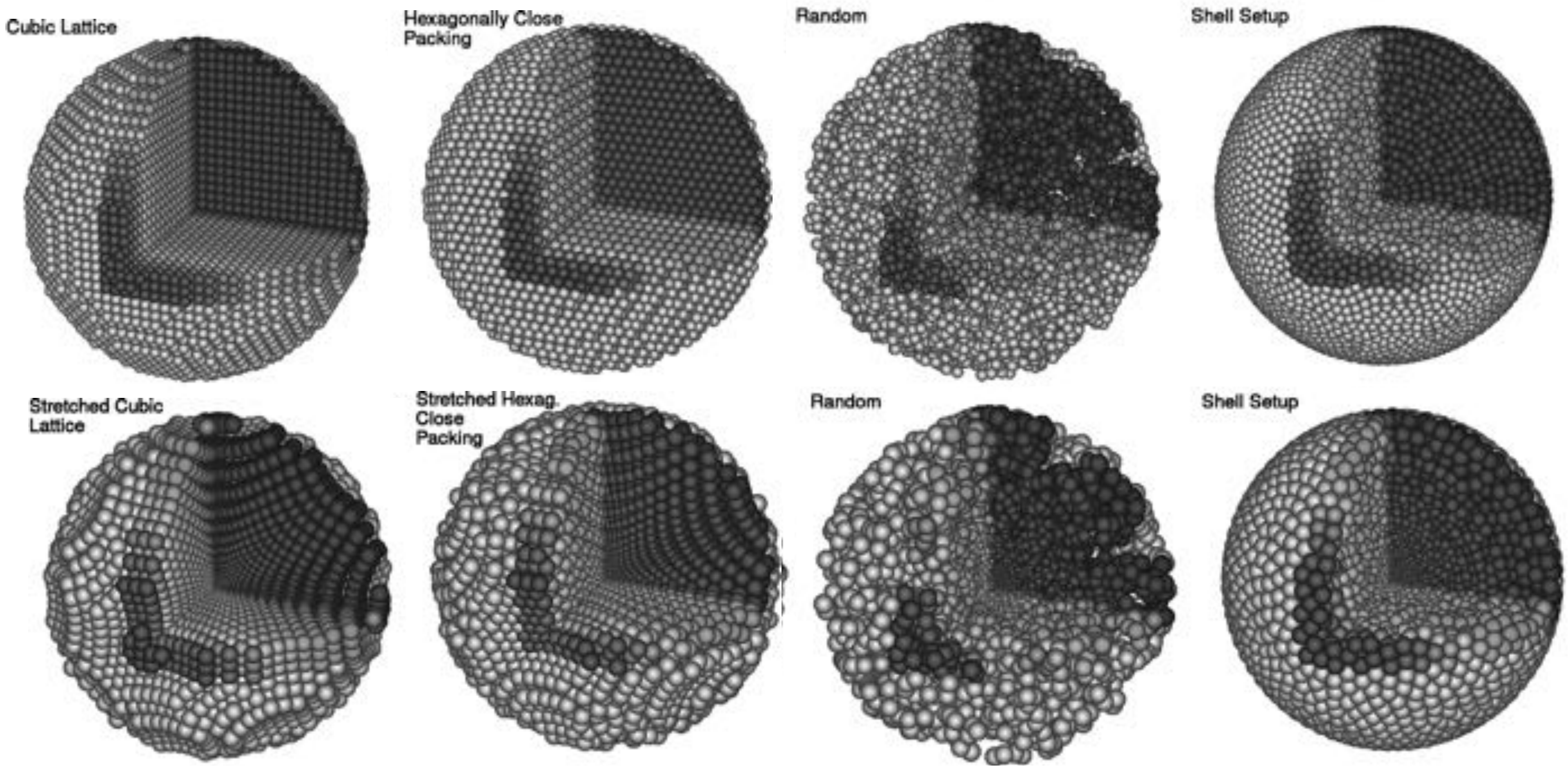
- Given a domain, how do we divide it up?
  - Lagrangian particles




- Each particle has a fixed mass
- Characteristics are calculated at the particles' locations

# *SPH vs Grid: Dividing the domain: SPH*

- How do we distribute the initial particles for uniform density (top)? For centrally condensed (bottom)? Does it matter (Morris, 1996)?

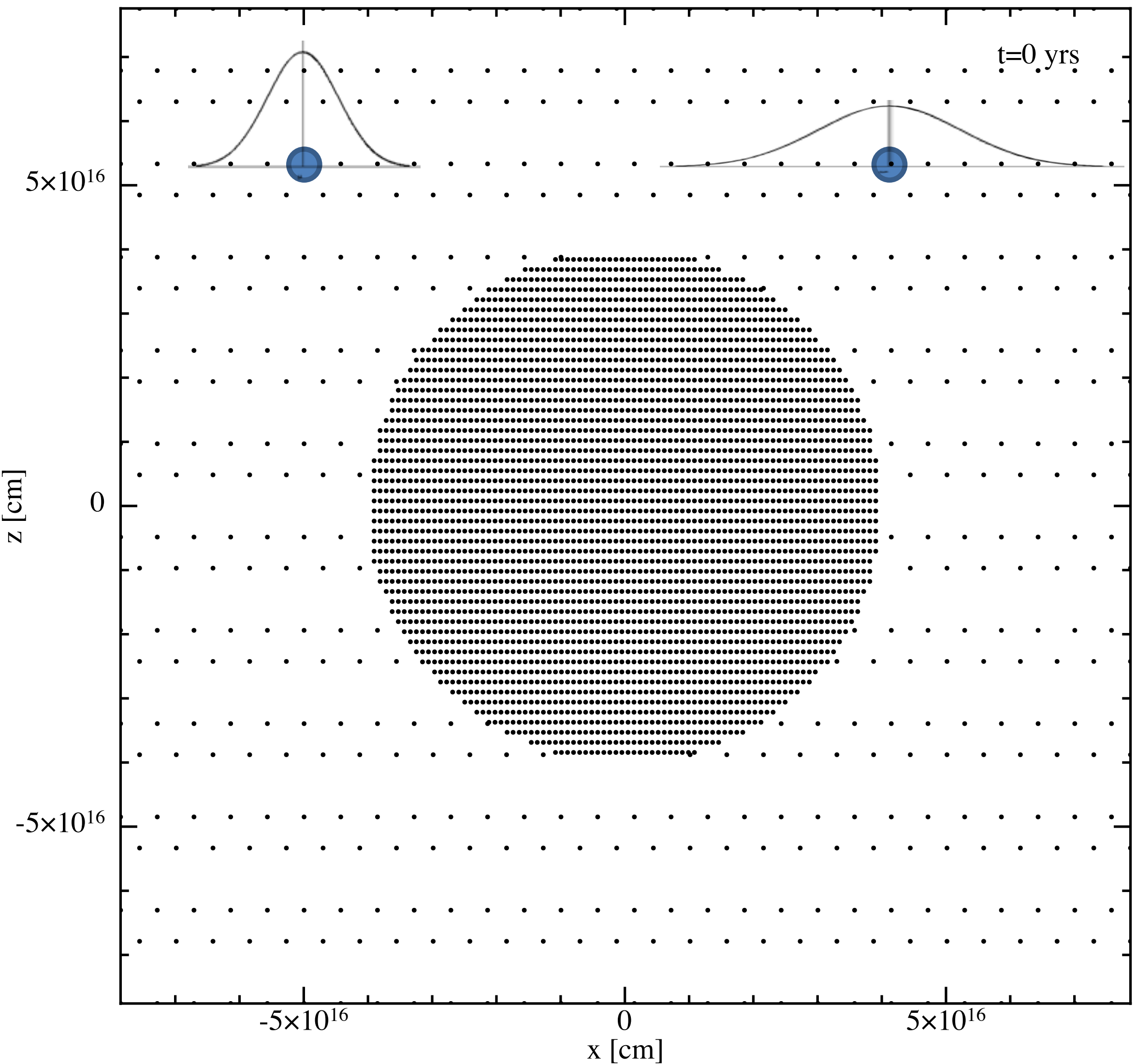





Diehl, Rockefeller, Fryer, Riethmiller & Statler (2015); particle size represents spacing/volume

# *SPH: Density*

- How is density calculated?




- Density summation

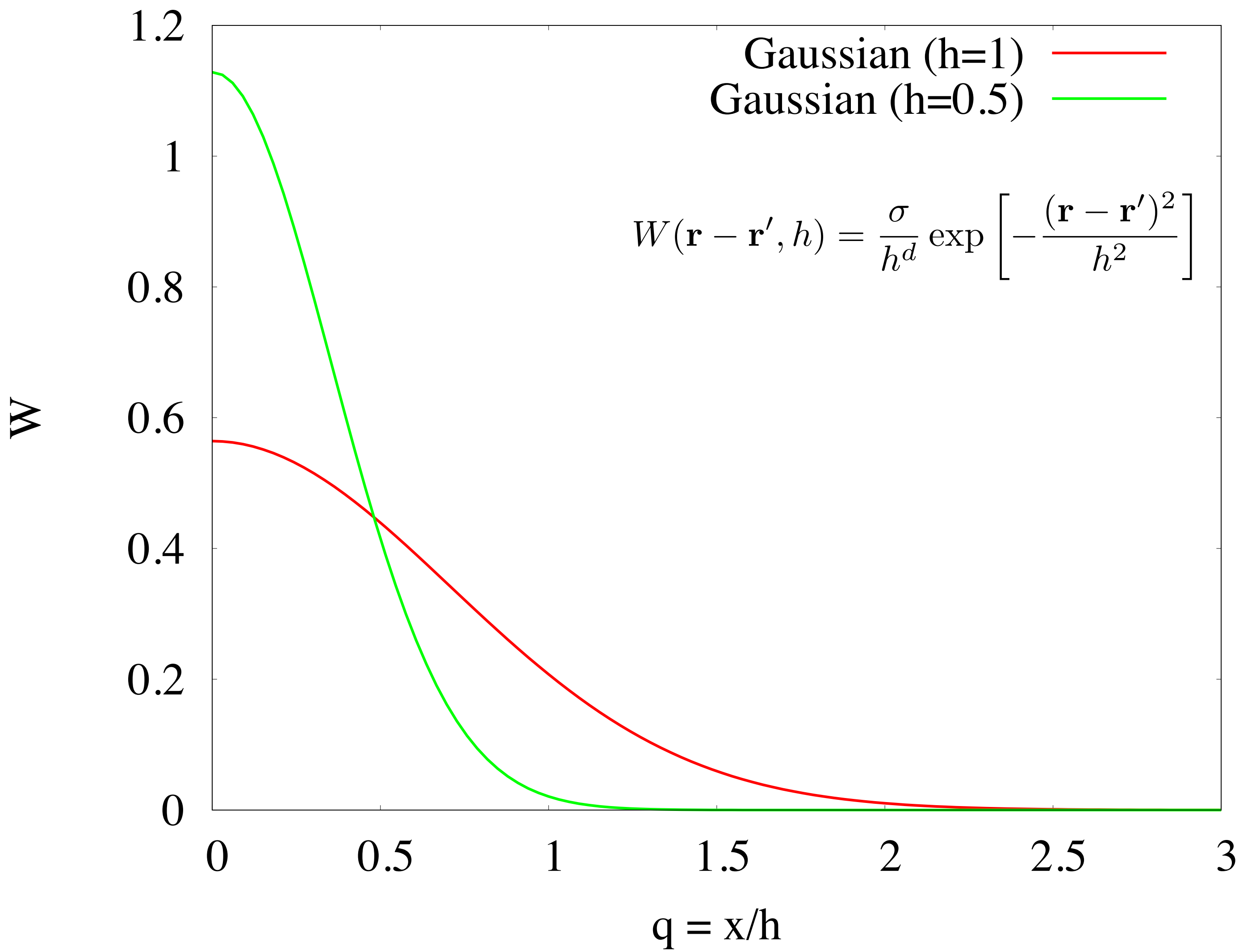
$$\rho(\mathbf{r}) = \sum_{b=1}^{N_{neigh}} m_b W(\mathbf{r} - \mathbf{r}_b, h),$$

- Where
  - $N$ is number of neighbours
  - $m_b$ is particle mass
  - $W$ is smoothing kernel
- Simplest kernel is a Gaussian:

$$W(\mathbf{r} - \mathbf{r}', h) = \frac{\sigma}{h^d} \exp\left[-\frac{(\mathbf{r} - \mathbf{r}')^2}{h^2}\right]$$

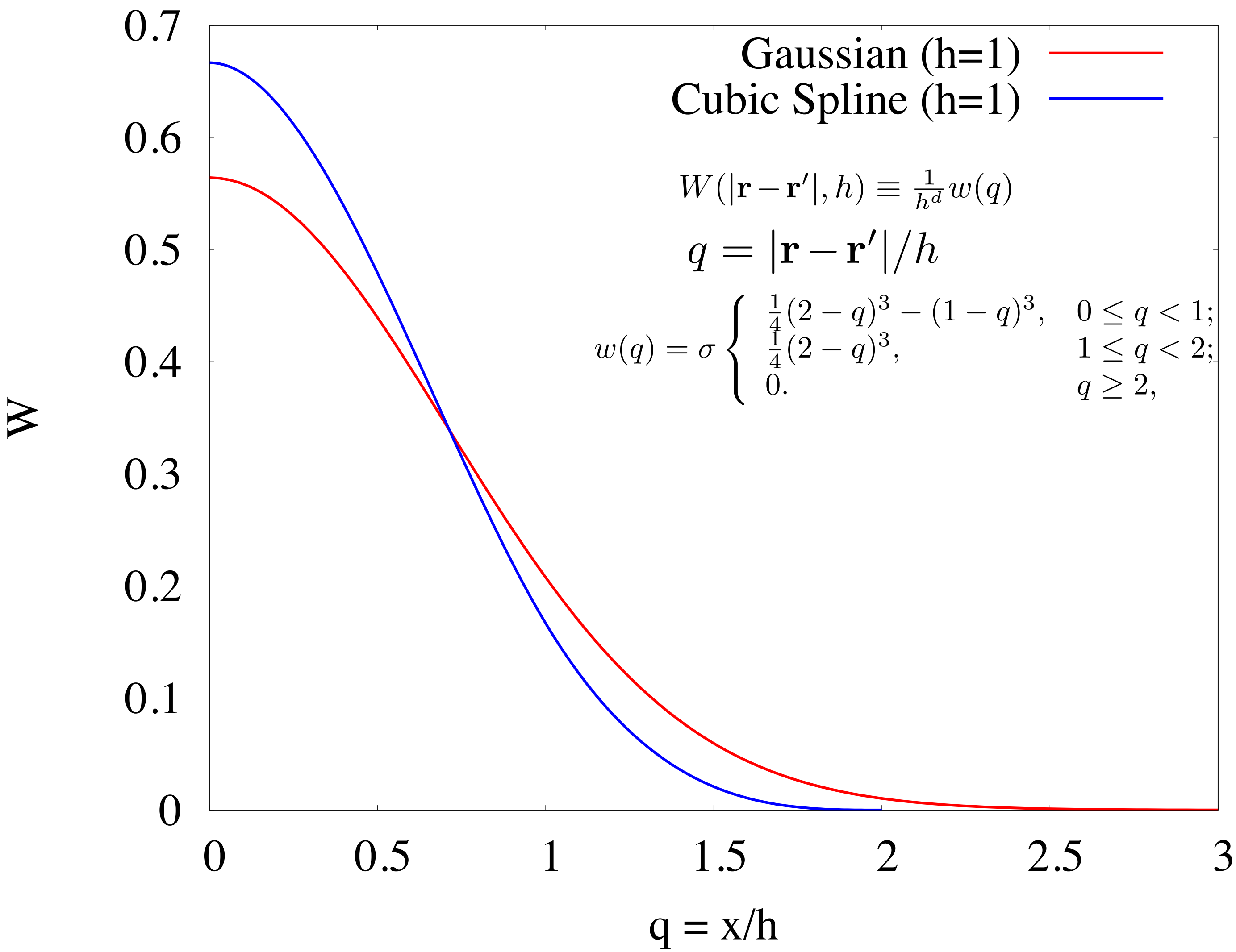
# SPH: Density: smoothing kernel

- How is density calculated?



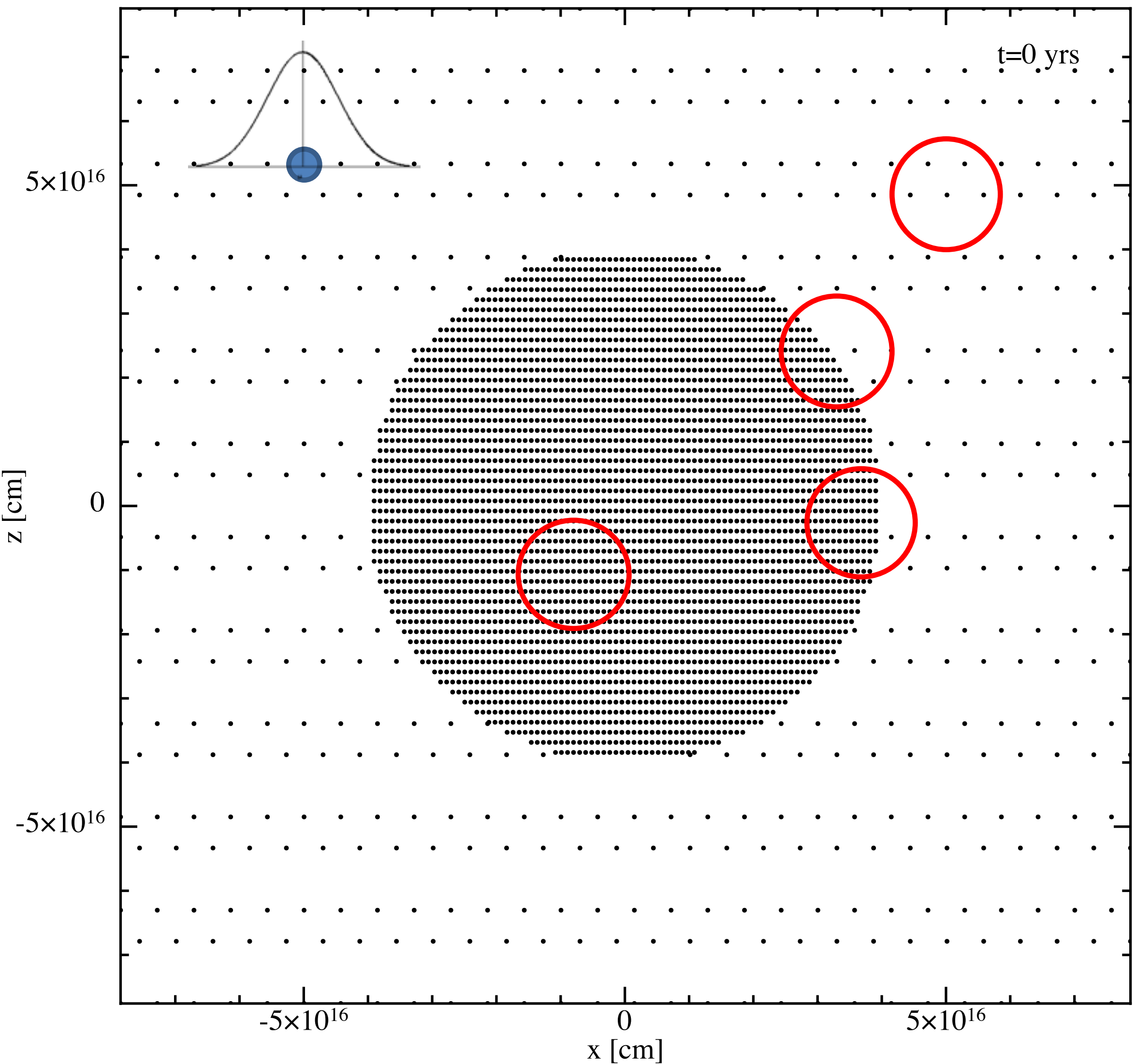
# *SPH: Density: Smoothing kernel*

- How is density calculated?



# *SPH: Density: Determining* h

- How is density calculated?

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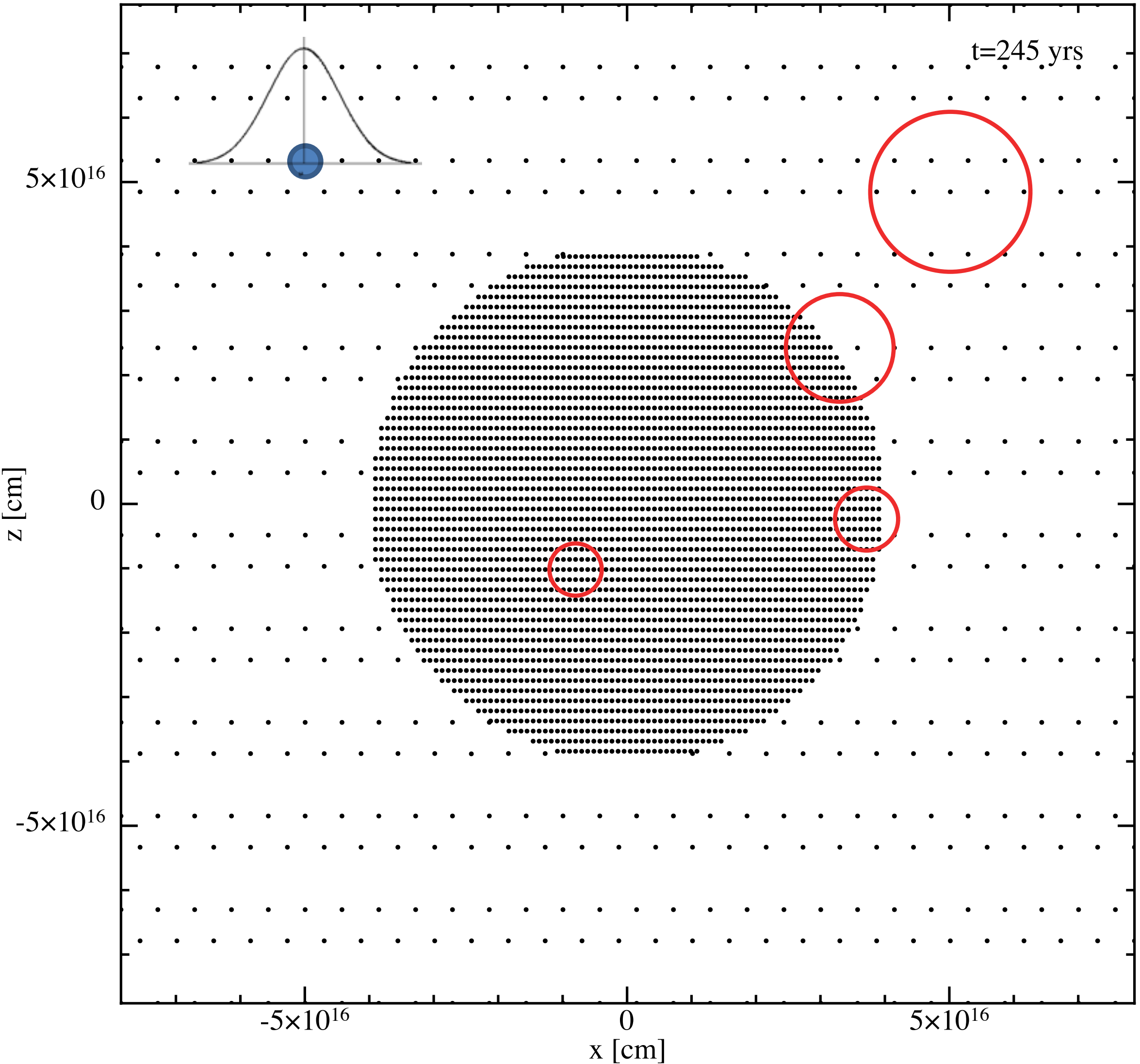
- What is $h$?

# SPH: Density: Determining h

- How is density calculated?

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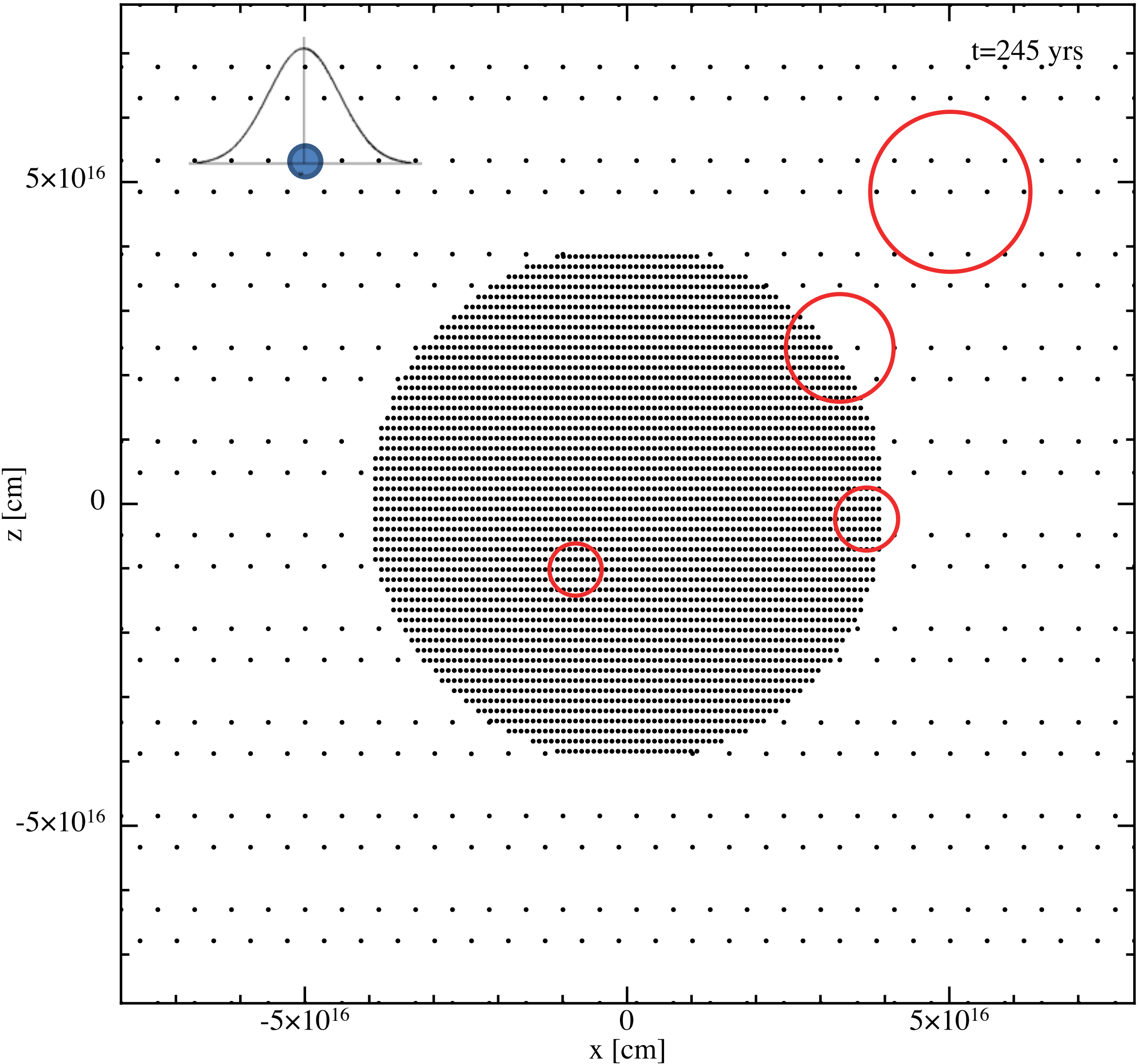
# *SPH: Density: Determining* h

- How is density calculated?




- Density summation

$$\rho_a = \sum_b m_b W_{ab}(\boldsymbol{r}_a - \boldsymbol{r}_b, h_a)$$

- Smoothing length relation

$$h_a = \eta \left(\frac{m_a}{\rho_a}\right)^{1/3}$$

- These equations must be iteratively solved

- For a cubic spline in 3D, there will be $N_{neigh} \sim 57$

# SPH: Continuum Equations

- Continuum Equations:
  - Continuity Equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\nabla\cdot\boldsymbol{v}$$

  - Equation of Motion

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{dt}} = -\frac{1}{\rho}\nabla P$$

  - Energy Equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{P}{\rho}\nabla\cdot\boldsymbol{v}$$

  - Equation of state (e.g.)

$$P = (\gamma - 1)\,\rho u$$

# SPH: Discrete Equations

- Discrete Equations:
  - Density Equation

$$\rho_a = \sum_b m_b W_{ab}(h_a); \quad h_a = \eta \left( \frac{m_a}{\rho_a} \right)^{1/3}$$

  - Equation of motion

$$\frac{\mathrm{d}v_a^i}{\mathrm{d}t} = \sum_b m_b \left[ \frac{P_a}{\Omega_a \rho_a^2} \nabla_a^i W_{ab}(h_a) + \frac{P_b}{\Omega_b \rho_b^2} \nabla_a^i W_{ab}(h_b) \right]$$

  - Energy Equation

$$\frac{\mathrm{d}u_a}{\mathrm{d}t} = \frac{P_a}{\Omega_a \rho_a^2} \sum_b m_b v_{ab}^i \nabla_a^i W_{ab}(h_a)$$

  - Equation of state (e.g.)

$$P_a = (\gamma - 1)\, \rho_a u_a$$

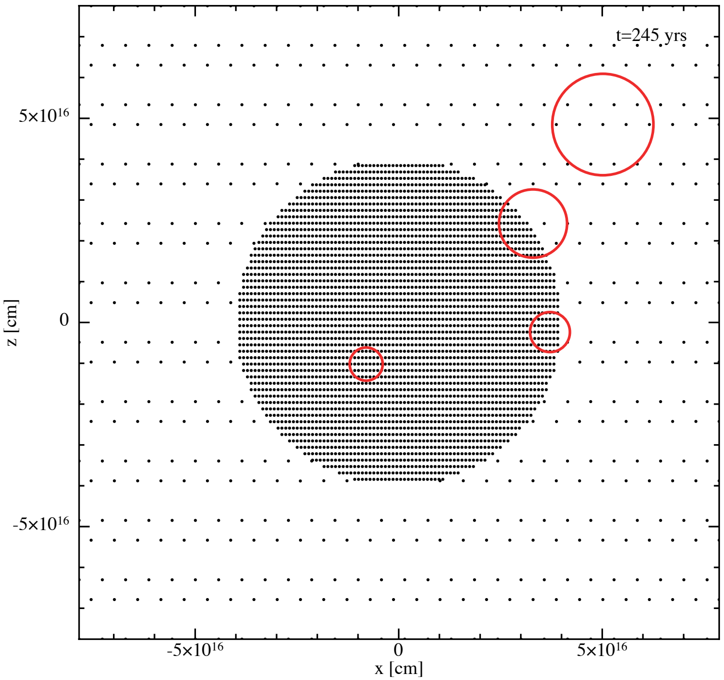
  - where

$$W_{ab}(h_a) \equiv W_{ab}(\boldsymbol{r}_a - \boldsymbol{r}_b, h_a); \quad v_{ab}^i \equiv v_a^i - v_b^i$$

For conversions from continuum to discrete,
see (e.g.) Monaghan (1992, 2005), Springel (2010), Price (2012)

# SPH: Pseudo-Code

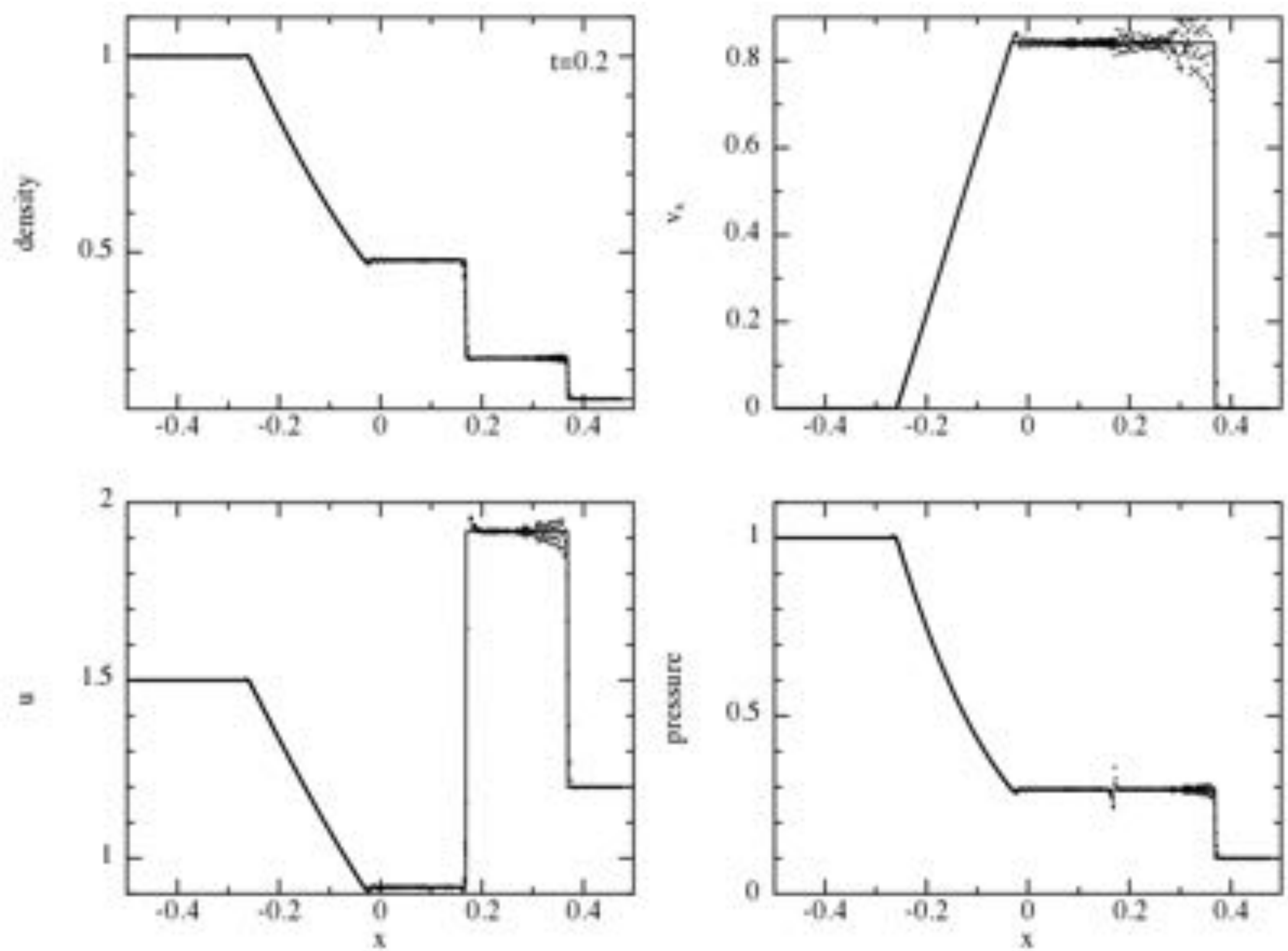
- Pseudo-Code
  - SPH
    - Requires at most $2N^2 + N$ iterations per step
      - If $N$=$10^6$, then there are $2\text{x}10^{12}$ iterations per step
    - Using neighbour finding algorithms requires $2NN_{neigh} + N$ iterations per step
      - If $N$=$10^6$, then there are $1.2\text{x}10^8$ iterations per step




```
do i = 1,N
   do j = 1,N_neigh
      Using j, calculate density of i
   enddo
enddo
do i = 1,N
   do j = 1,N_neigh
      Using j, calculate forces of i
   enddo
enddo
do i = 1,N
   Using updated forces, determine
   new v, r & B of i
enddo
```

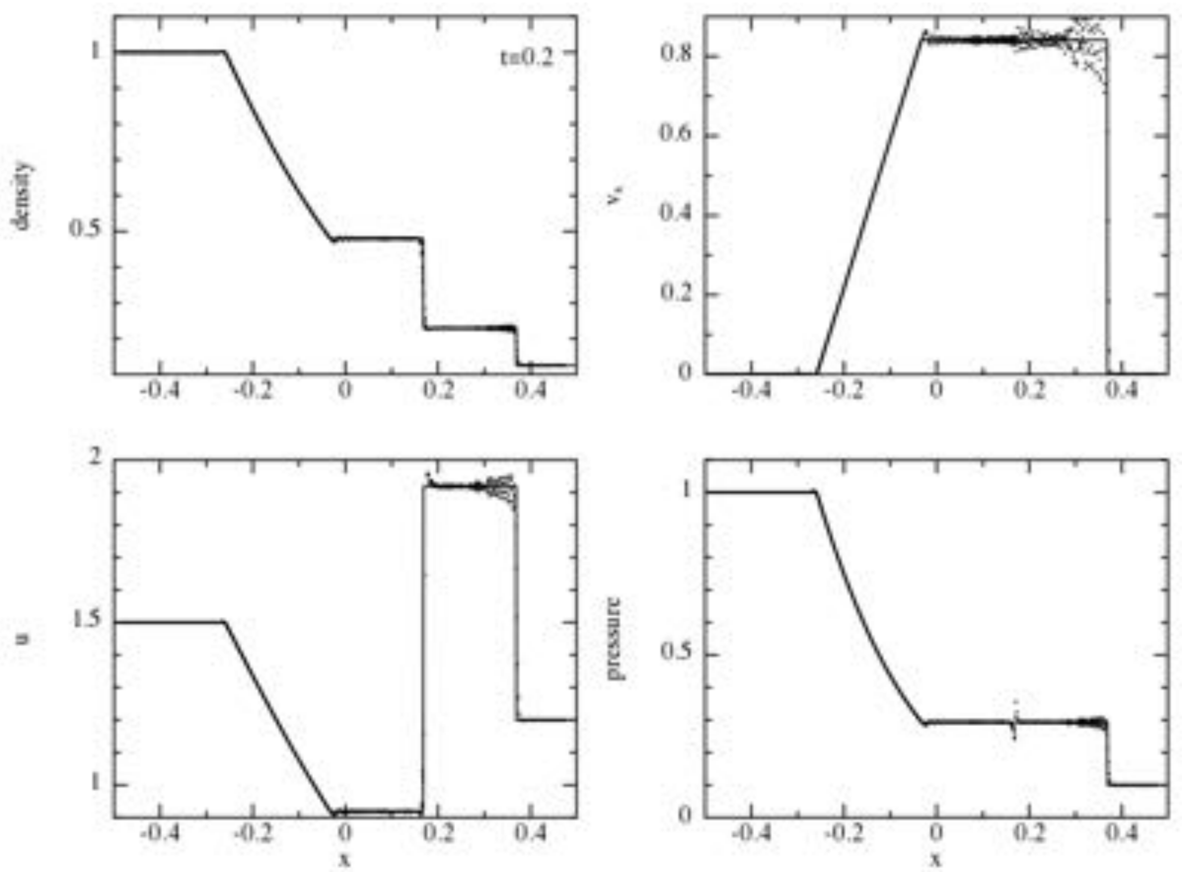
# *SPH: Sod Shock Tube*

- As written on the previous slide



# *SPH: Discrete Equations: Missing terms*

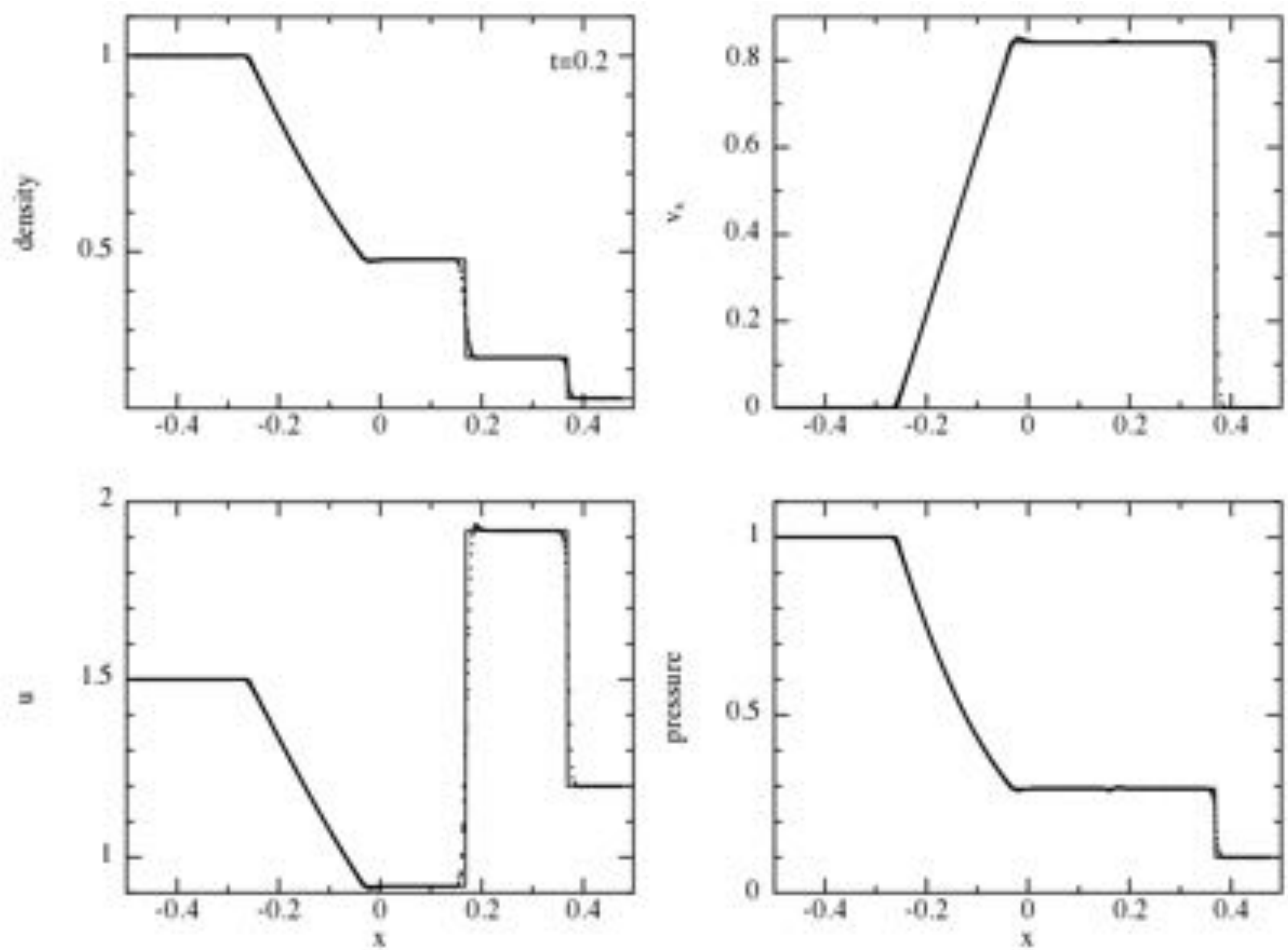
- Missing terms:
  - Equation of motion
    - Artificial viscosity
  - Energy Equation
    - Artificial conductivity

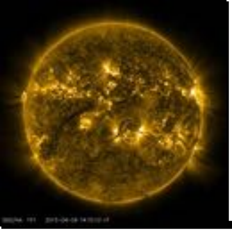



Various forms.
see (e.g.) Monaghan (1992, 2005), Springel (2010), Price (2012)

# SPH: Sod Shock Tube

- Maximal artificial viscosity and conductivity





# *SPMHD: Discrete Equations*

- SPH equation of motion is derived via a the Lagrangian (Price & Monaghan, 2004)

$$L_{\mathrm{MHD}} = \sum_b m_b \left[ \frac{1}{2} v_b^2 - u_b(\rho_b, s_b) - \frac{1}{2\mu_0} \frac{B_b^2}{\rho_b} \right]$$

- SPMHD equation of motion requires the variational principle of $\int \delta L dt = 0$, where

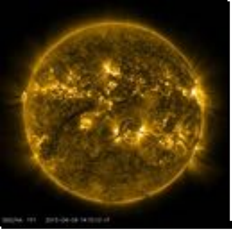
$$\delta L = m_a \mathbf{v}_a \cdot \delta \mathbf{v}_a - \sum_b m_b \left[ \left. \frac{\partial u_b}{\partial \rho_b} \right|_s \delta\rho_b + \frac{1}{2\mu_0} \left( \frac{B_b}{\rho_b} \right)^2 \delta\rho_b + \frac{1}{\mu_0} \mathbf{B}_b \cdot \delta \left( \frac{\mathbf{B}_b}{\rho_b} \right) \right]$$

- After some math....

$$\frac{\mathrm{d}v_a^i}{\mathrm{d}t} = \sum_b m_b \left[ \frac{S_a^{ij}}{\Omega_a \rho_a^2} \nabla_a^j W_{ab}(h_a) + \frac{S_b^{ij}}{\Omega_b \rho_b^2} \nabla_a^j W_{ab}(h_b) \right]$$

$$S_a^{ij} \equiv -\left( P_a + \frac{1}{2\mu_0} B_a^2 \right) \delta^{ij} + \frac{1}{\mu_0} B_a^i B_a^j$$

Section 7 of Price (2012)



# SPMHD: Discrete Equations

- Discrete Equations:
  - Density Equation

$$\rho_a = \sum_b m_b W_{ab}(h_a); \quad h_a = \eta \left(\frac{m_a}{\rho_a}\right)^{1/3}$$

  - Equation of Motion

$$\frac{\mathrm{d}v_a^i}{\mathrm{d}t} = \sum_b m_b \left[\frac{S_a^{ij}}{\Omega_a \rho_a^2} \nabla_a^j W_{ab}(h_a) + \frac{S_b^{ij}}{\Omega_b \rho_b^2} \nabla_a^j W_{ab}(h_b)\right]$$

  - Induction Equation

$$\frac{\mathrm{d}B_a^i}{\mathrm{d}t} = -\frac{1}{\Omega_a \rho_a} \sum_b m_b \left[v_{ab}^i B_a^j \nabla_a^j W_{ab}\left(h_a\right) - B_a^i v_{ab}^j \nabla_a^j W_{ab}\left(h_a\right)\right]$$

  - Energy Equation

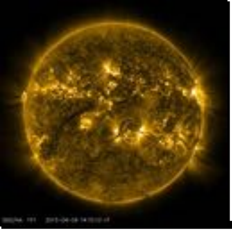
$$\frac{\mathrm{d}u_a}{\mathrm{d}t} = \frac{P_a}{\Omega_a \rho_a^2} \sum_b m_b v_{ab}^i \nabla_a^i W_{ab}(h_a)$$

  - MHD stress tensor

$$S_a^{ij} \equiv -\left(P_a + \frac{1}{2\mu_0} B_a^2\right) \delta^{ij} + \frac{1}{\mu_0} B_a^i B_a^j$$



  - Note: In all SPMHD equations, ***B*** has been normalised such that $\boldsymbol{B} = \boldsymbol{B}/\sqrt{\mu_0}$



# SPMHD: Continuum Equations

- Continuum Equations:
  - Continuity Equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\nabla\cdot\boldsymbol{v}$$

  - Equation of Motion

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{dt}} = -\frac{1}{\rho}\nabla\left[\left(P+\frac{B^2}{2\mu_0}\right)I-\frac{1}{\mu_0}\boldsymbol{B}\boldsymbol{B}\right]$$

  - Induction Equation

$$\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} = (\boldsymbol{B}\cdot\boldsymbol{\nabla})\,\boldsymbol{v}-\boldsymbol{B}\,(\boldsymbol{\nabla}\cdot\boldsymbol{v})$$

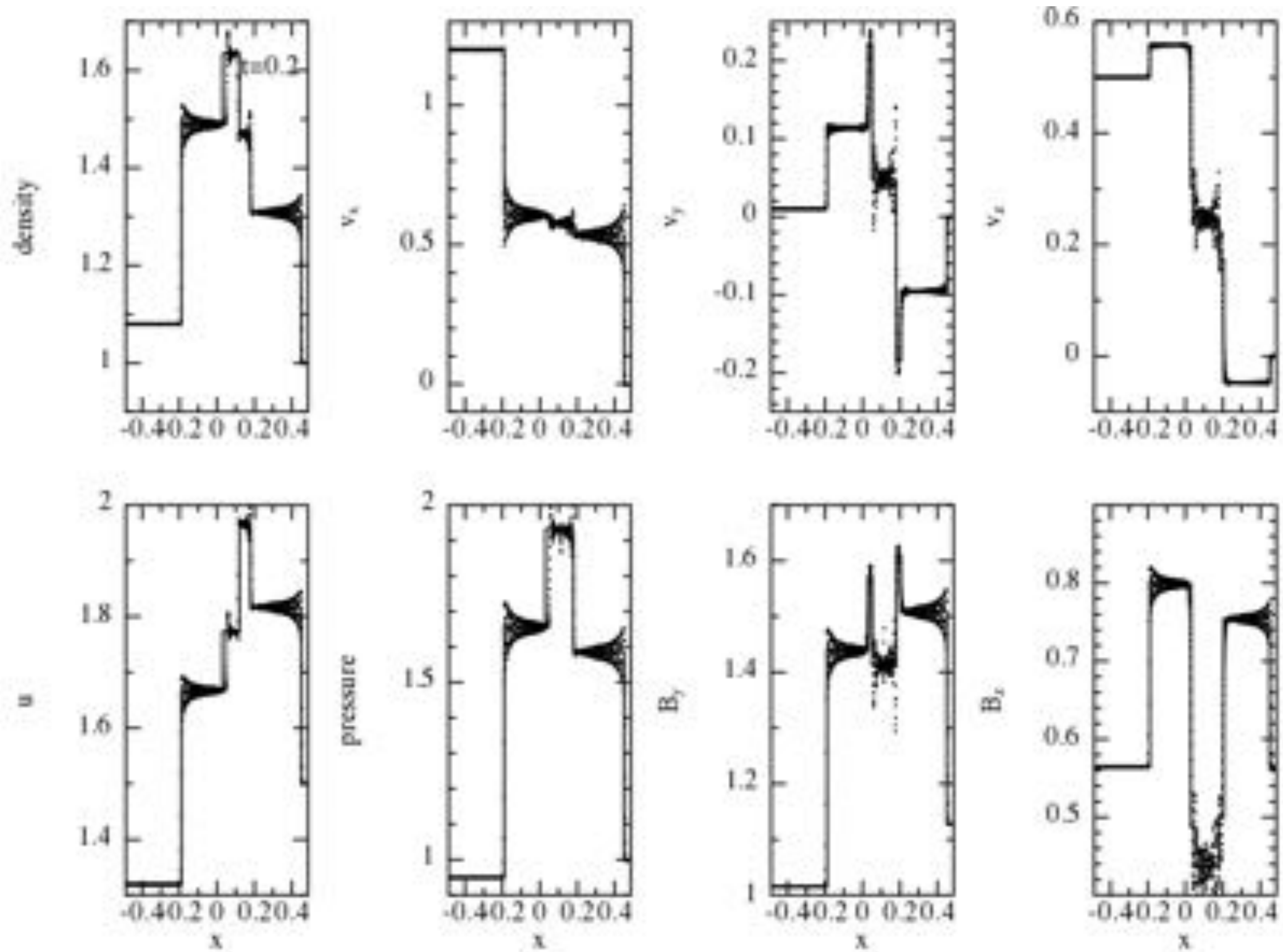
  - Energy Equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{P}{\rho}\nabla\cdot\boldsymbol{v}$$

  - Equation of state

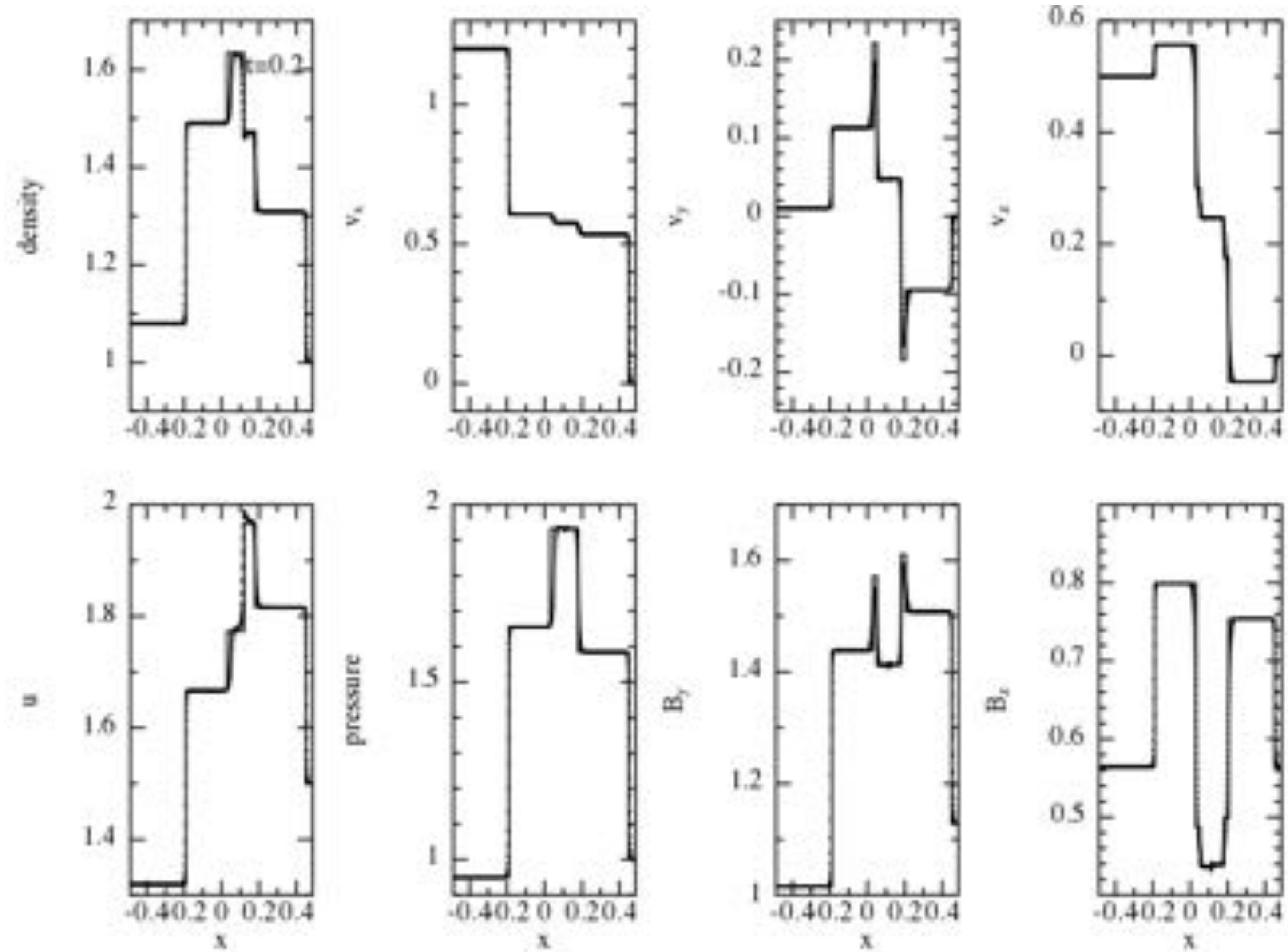
# *SPMHD: Ryu-Jones Shock Tube*

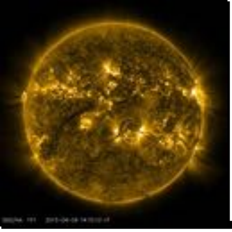
- As written on the previous slide



# *SPMHD: Ryu-Jones Shock Tube*

- With maximal artificial viscosity, conductivity and resistivity



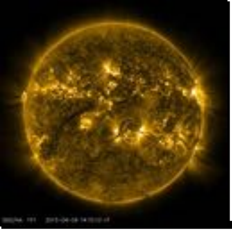


# SPMHD: Artificial Viscosity

- Artificial viscosity

$$
\begin{aligned}
\frac{\mathrm{d}v_a^i}{\mathrm{d}t} &= \sum_b m_b \left[ \frac{S_a^{ij} + q_a^{ab}}{\Omega_a \rho_a^2} \nabla_a^j W_{ab}(h_a) + \frac{S_b^{ij} + q_b^{ab}}{\Omega_b \rho_b^2} \nabla_a^j W_{ab}(h_b) \right] \\
q_a^{ab} &= \begin{cases} -\frac{1}{2}\rho_a v_{\mathrm{sig},a} \boldsymbol{v}_{ab} \cdot \hat{\boldsymbol{r}}_{ab}; & \boldsymbol{v}_{ab} \cdot \hat{\boldsymbol{r}}_{ab} < 0 \\ 0; & \text{else} \end{cases} \\
v_{\mathrm{sig},a} &= \alpha_a^{\mathrm{AV}} \sqrt{c_{\mathrm{s},a}^2 + v_{\mathrm{A},a}^2} + \beta^{\mathrm{AV}} \left| \boldsymbol{v}_{ab} \cdot \hat{\boldsymbol{r}}_{ab} \right| \\
\alpha_a^{\mathrm{AV}} &\in [0,1] \quad \text{Calculated using (e.g.) the Cullen \& Dehnen (2010) switch} \\
\beta^{\mathrm{AV}} &= 2
\end{aligned}
$$

  - Applied only to shocks

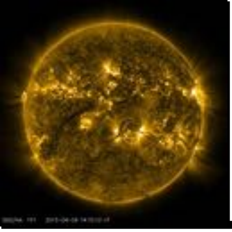


# SPMHD: Artificial Resistivity

- Artificial resistivity (Tricco & Price, 2013)

$$
\begin{aligned}
\frac{\mathrm{d}B_a^i}{\mathrm{d}t} &= -\frac{1}{\Omega_a\rho_a}\sum_b m_b\left[v_{ab}^i B_a^j \nabla_a^j W_{ab}\left(h_a\right) - B_a^i v_{ab}^j \nabla_a^j W_{ab}\left(h_a\right)\right] + \left.\frac{\mathrm{d}B_a^i}{\mathrm{d}t}\right|_{\mathrm{art}} \\
\left.\frac{\mathrm{d}B_a^i}{\mathrm{d}t}\right|_{\mathrm{art}} &= \frac{\rho_a}{2}\sum_b m_b B_{ab}^i\left[\frac{\alpha_a^{\mathrm{B}} v_{\mathrm{sig},a}\hat{r}_{ab}^j \nabla_a^j W_{ab}(h_a)}{\Omega_a\rho_a^2} + \frac{\alpha_b^{\mathrm{B}} v_{\mathrm{sig},b}\hat{r}_{ab}^j \nabla_a^j W_{ab}(h_b)}{\Omega_b\rho_b^2}\right] \\
v_{ab}^i &= v_a^i - v_b^i \\
B_{ab}^i &= B_a^i - B_b^i \\
v_{\mathrm{sig},a} &= \sqrt{c_{\mathrm{s},a}^2 + v_{\mathrm{A},a}^2} \\
\alpha_a^{\mathrm{B}} &= \min\left(\frac{h_a\left|\nabla \boldsymbol{B}_a\right|}{\left|\boldsymbol{B}_a\right|}, 1\right) \\
\left|\nabla \boldsymbol{B}_a\right| &\equiv \sqrt{\sum_i\sum_j\left|\frac{\partial B_a^i}{\partial x_a^j}\right|^2}
\end{aligned}
$$

- Always applied if there is a gradient in the magnetic field (i.e. $|\nabla \mathbf{B}| > 0$)



# SPMHD: Artificial Resistivity

- Artificial resistivity (Price, et al, submitted)

$$\frac{\mathrm{d}B_a^i}{\mathrm{d}t} = -\frac{1}{\Omega_a\rho_a}\sum_b m_b\left[v_{ab}^i B_a^j \nabla_a^j W_{ab}\left(h_a\right) - B_a^i v_{ab}^j \nabla_a^j W_{ab}\left(h_a\right)\right] + \left.\frac{\mathrm{d}B_a^i}{\mathrm{d}t}\right|_{\mathrm{art}}$$

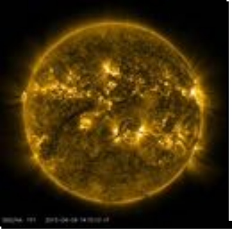
$$\left.\frac{\mathrm{d}B_a^i}{\mathrm{d}t}\right|_{\mathrm{art}} = \frac{\rho_a}{2}\sum_b m_b \alpha^{\mathrm{B}} v_{\mathrm{sig},ab} B_{ab}^i \left[\frac{\hat{r}_{ab}^j \nabla_a^j W_{ab}(h_a)}{\Omega_a\rho_a^2} + \frac{\hat{r}_{ab}^j \nabla_a^j W_{ab}(h_b)}{\Omega_b\rho_b^2}\right]$$

$$B_{ab}^i = B_a^i - B_b^i$$

$$v_{\mathrm{sig},ab} = \left|\boldsymbol{v}_{ab}\times\hat{\boldsymbol{r}}_{ab}\right|$$

$$\alpha^{\mathrm{B}} \equiv 1$$

  - Always applied for non-zero velocity
  - Less resistive that that from Tricco & Price (2013)



# SPMHD: Artificial Resistivity

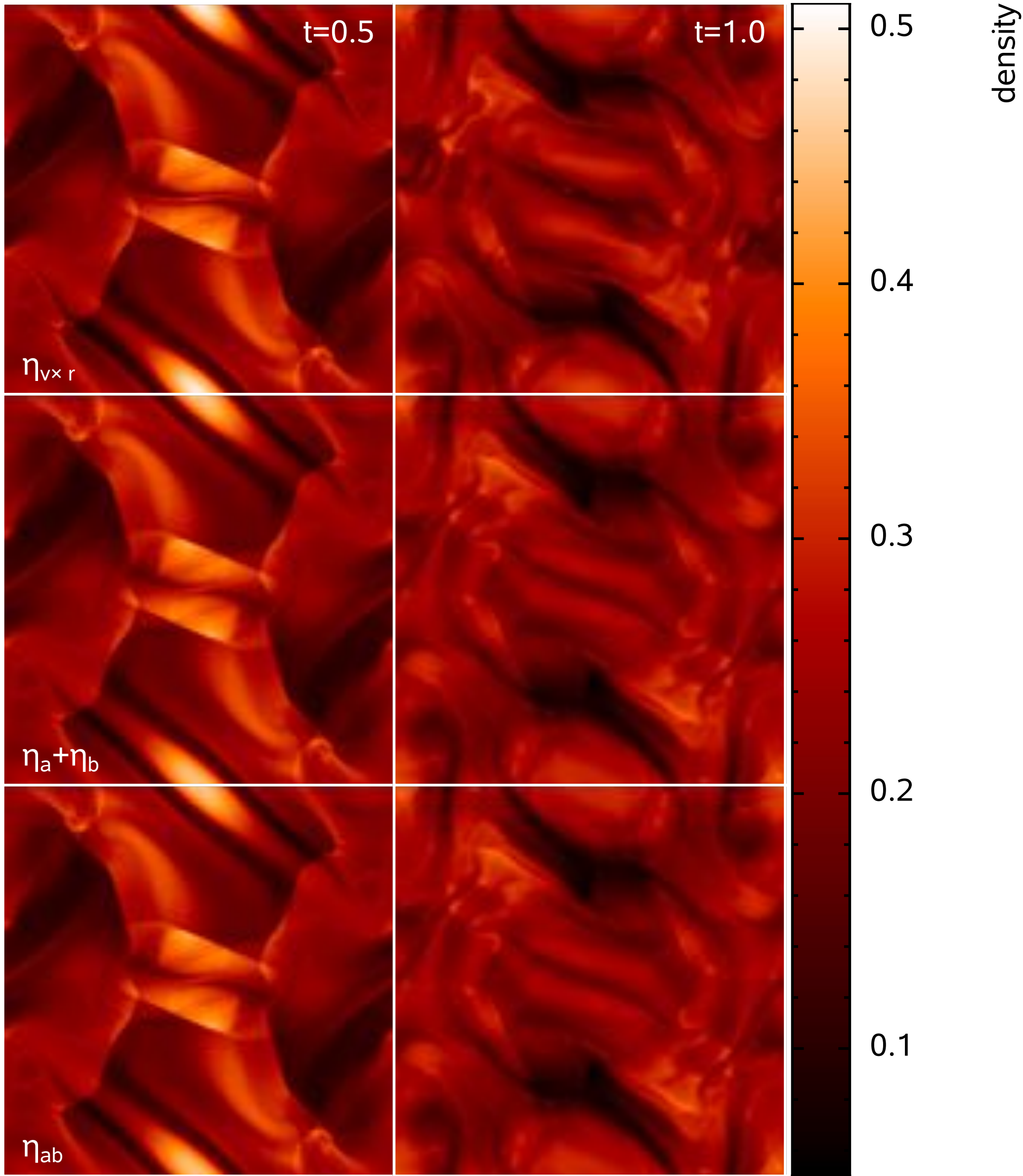
- Price et. al. (2017) artificial resistivity

$$v_{\mathrm{sig},ab} = |\boldsymbol{v}_{ab} \times \hat{\boldsymbol{r}}_{ab}|$$
$$\alpha^{\mathrm{B}} \equiv 1$$

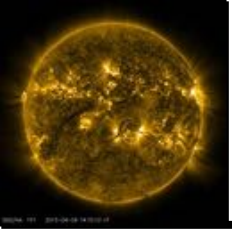
- Tricco & Price (2013)

$$v_{\mathrm{sig},a} = \sqrt{c_{\mathrm{s},a}^2 + v_{\mathrm{A},a}^2}$$
$$\alpha_a^{\mathrm{B}} = \min\left(\frac{h_a\,|\nabla \boldsymbol{B}_a|}{|\boldsymbol{B}_a|}, 1\right)$$

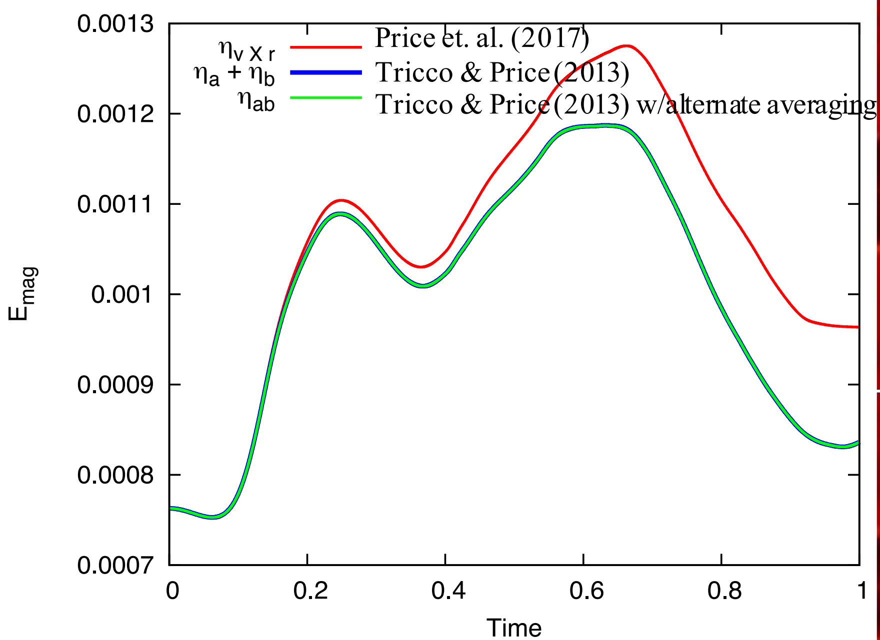
- Tricco & Price (2013) with alternate averaging

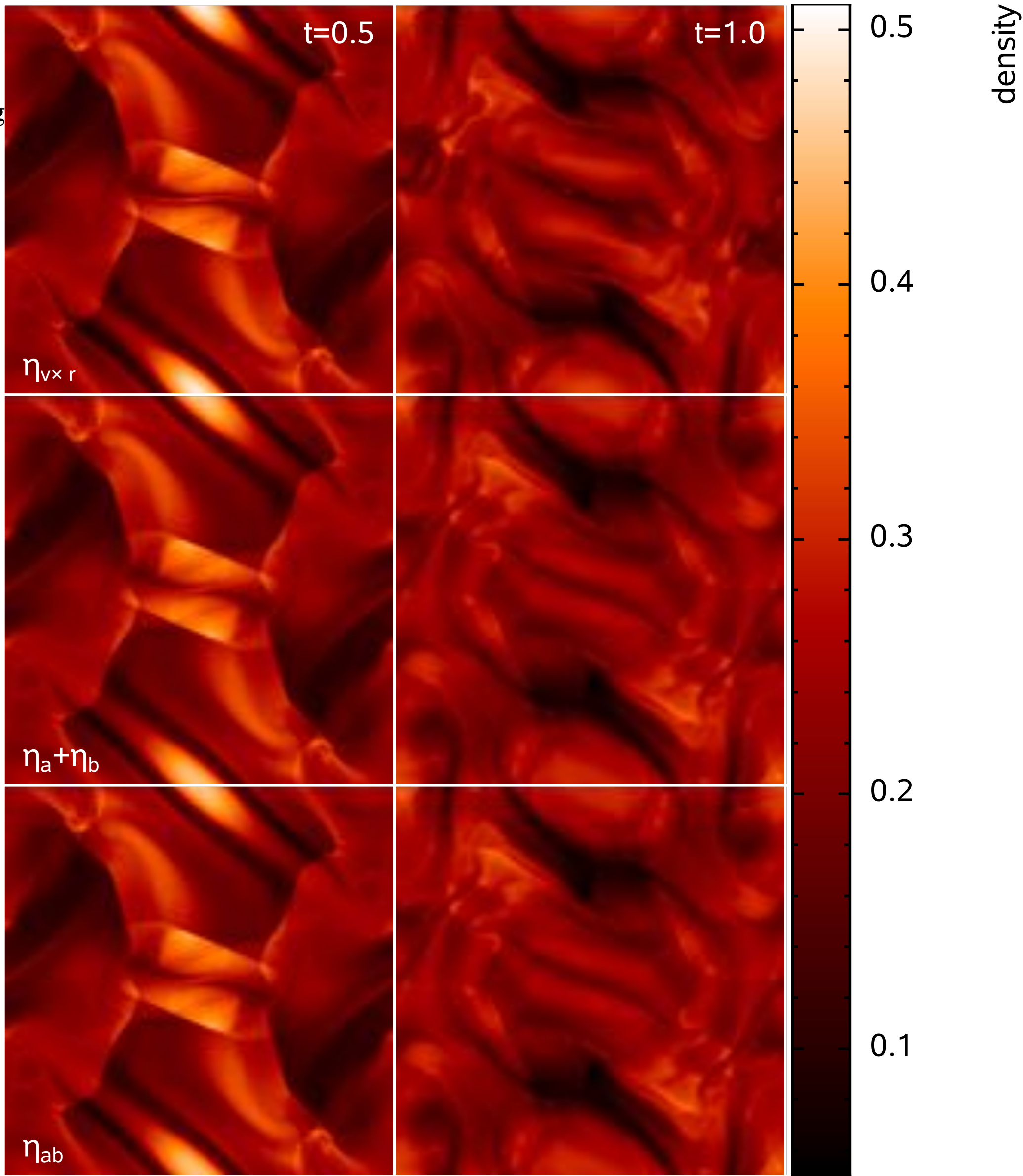



Wurster, Bate, Price & Tricco (2017)

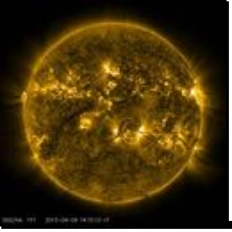


# *SPMHD: Artificial Resistivity*



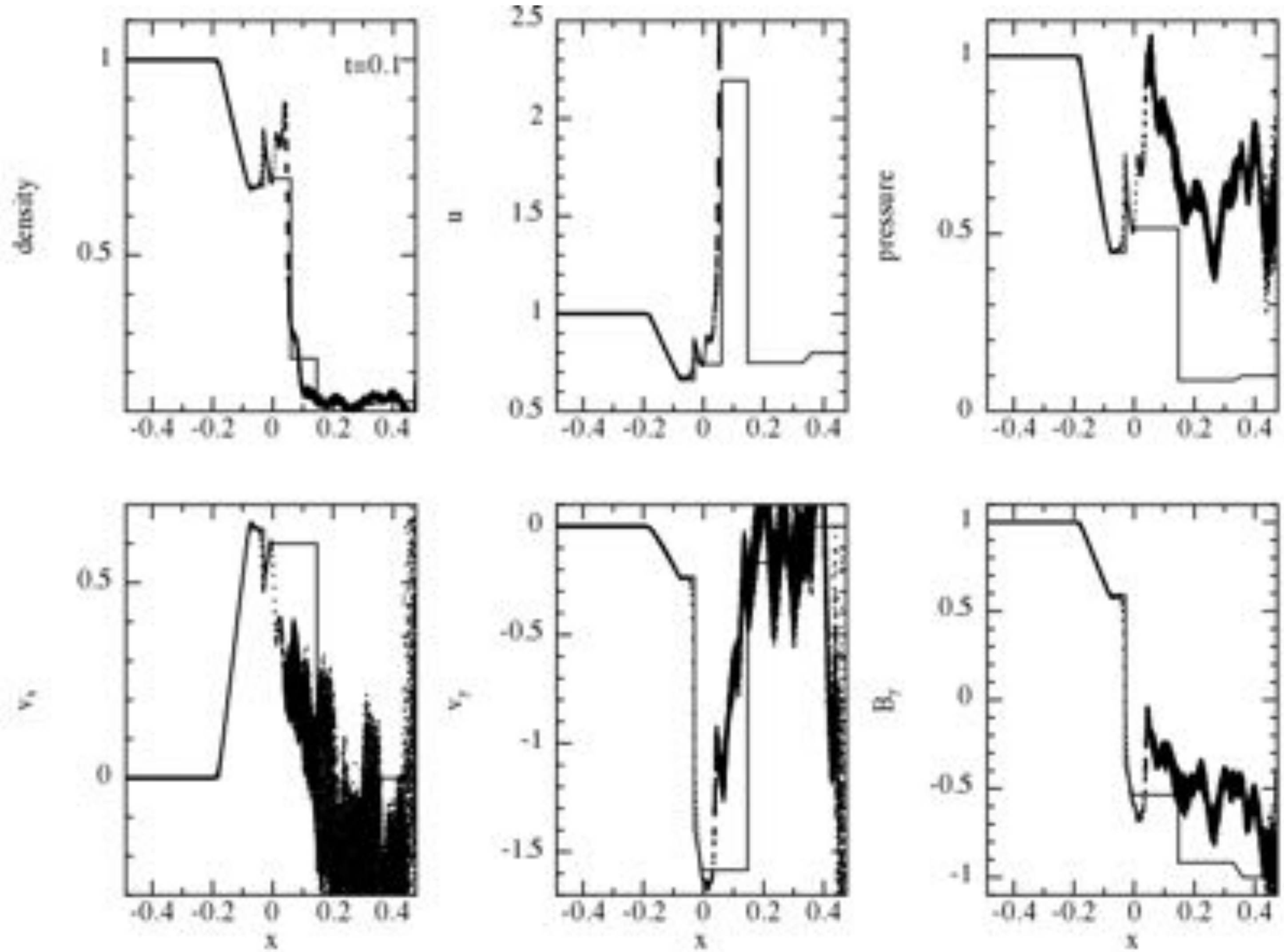



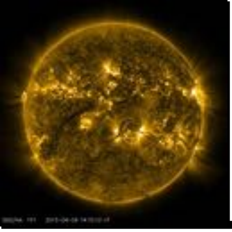

Wurster, Bate, Price & Tricco (2017)



# *SPMHD: Brio-Wu Shock Tube*

- With maximal artificial viscosity, conductivity and resistivity





# SPMHD: Discrete Equations

- Discrete Equations:
  - Density Equation

$$\rho_a = \sum_b m_b W_{ab}(h_a); \quad h_a = \eta \left(\frac{m_a}{\rho_a}\right)^{1/3}$$

  - Momentum Equation

$$\frac{\mathrm{d}v_a^i}{\mathrm{d}t} = \sum_b m_b \left[\frac{S_a^{ij} + q_a^{ab}}{\Omega_a \rho_a^2} \nabla_a^j W_{ab}(h_a) + \frac{S_b^{ij} + q_b^{ab}}{\Omega_b \rho_b^2} \nabla_a^j W_{ab}(h_b)\right]$$

  - Induction Equation

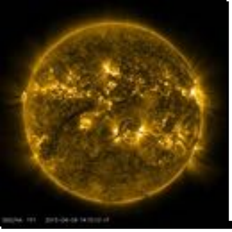
$$\frac{\mathrm{d}B_a^i}{\mathrm{d}t} = -\frac{1}{\Omega_a \rho_a} \sum_b m_b \left[v_{ab}^i B_a^j \nabla_a^j W_{ab}\left(h_a\right) - B_a^i v_{ab}^j \nabla_a^j W_{ab}\left(h_a\right)\right] + \left.\frac{\mathrm{d}B_a^i}{\mathrm{d}t}\right|_{\mathrm{art}}$$

  - Energy Equation

$$\frac{\mathrm{d}u_a}{\mathrm{d}t} = \frac{P_a}{\Omega_a \rho_a^2} \sum_b m_b v_{ab}^i \nabla_a^i W_{ab}(h_a) + \left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{\mathrm{art}}$$

  - MHD stress tensor

$$S_a^{ij} \equiv -\left(P_a + \frac{1}{2\mu_0} B_a^2\right) \delta^{ij} + \frac{1}{\mu_0} B_a^i B_a^j$$



# *SPMHD: Discrete Equations*

**IMPLICITLY INCLUDES NUMERICAL MAGNETIC MONOPOLES!!!**

- Discrete Equations:
  - Density Equation

$$\rho_a = \sum_b m_b W_{ab}(h_a); \quad h_a = \eta \left(\frac{m_a}{\rho_a}\right)^{1/3}$$

  - Momentum Equation

$$\frac{\mathrm{d}v_a^i}{\mathrm{d}t} = \sum_b m_b \left[\frac{S_a^{ij} + q_a^{ab}}{\Omega_a \rho_a^2} \nabla_a^j W_{ab}(h_a) + \frac{S_b^{ij} + q_b^{ab}}{\Omega_b \rho_b^2} \nabla_a^j W_{ab}(h_b)\right]$$

  - Induction Equation

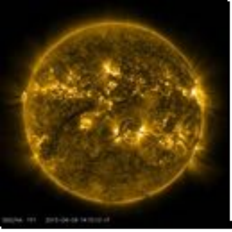
$$\frac{\mathrm{d}B_a^i}{\mathrm{d}t} = -\frac{1}{\Omega_a \rho_a} \sum_b m_b \left[v_{ab}^i B_a^j \nabla_a^j W_{ab}\left(h_a\right) - B_a^i v_{ab}^j \nabla_a^j W_{ab}\left(h_a\right)\right] + \left.\frac{\mathrm{d}B_a^i}{\mathrm{d}t}\right|_{\mathrm{art}}$$

  - Energy Equation

$$\frac{\mathrm{d}u_a}{\mathrm{d}t} = \frac{P_a}{\Omega_a \rho_a^2} \sum_b m_b v_{ab}^i \nabla_a^i W_{ab}(h_a) + \left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{\mathrm{art}}$$

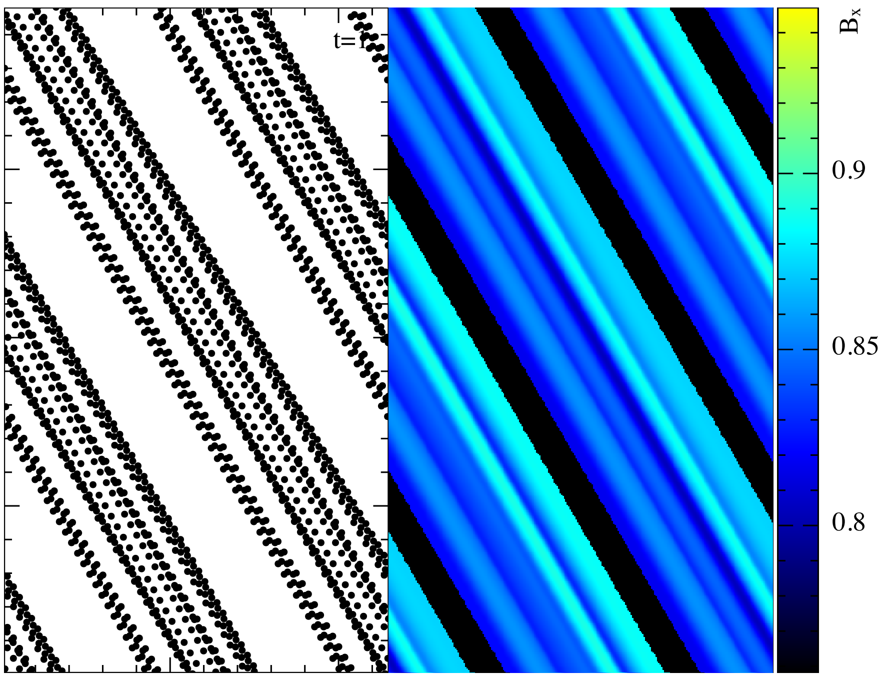
  - MHD stress tensor

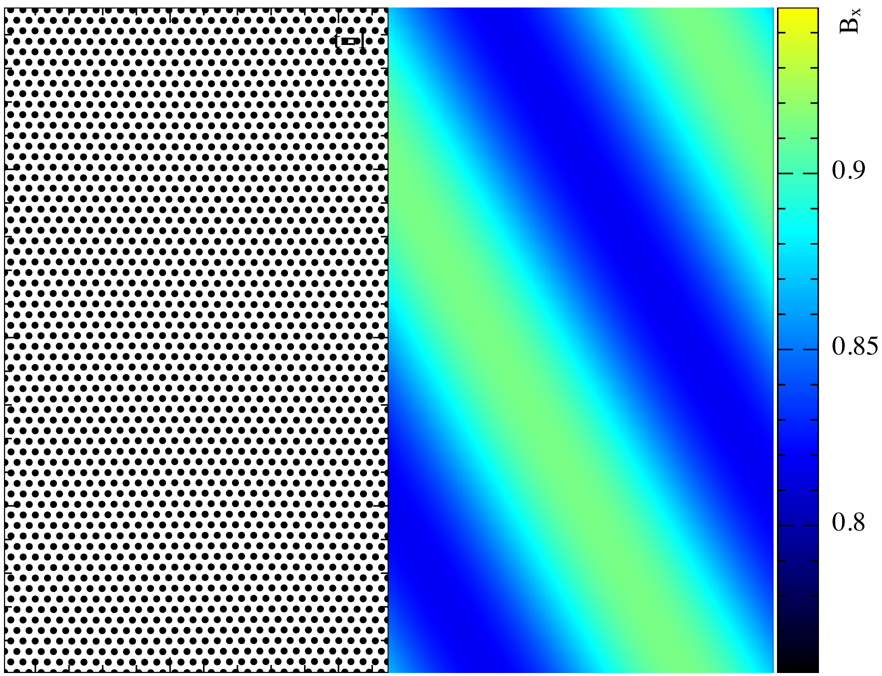
$$S_a^{ij} \equiv -\left(P_a + \frac{1}{2\mu_0} B_a^2\right) \delta^{ij} + \frac{1}{\mu_0} B_a^i B_a^j$$

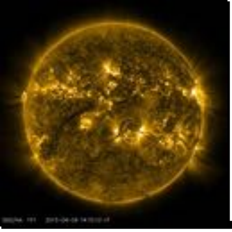


# *SPMHD: Tensile Instability*

- Tensile instability: the artificial clumping due to negative pressure (i.e. attractive forces)
- particle distribution & *x*-component of the magnetic field in the 2.5D circularly polarized Alfven wave test using the (unstable) conservative SPMHD force (left figure) and with a stable formulation (right figure), shown after 1 wave crossing time.





# SPMHD: Tensile Instability

- Momentum Equation (excluding artificial viscosity)

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla\left[\left(P + \frac{B^2}{2\mu_0}\right)I - \frac{1}{\mu_0}\boldsymbol{B}\boldsymbol{B}\right]$$

$$= -\frac{\nabla P}{\rho} - \frac{1}{\mu_0\rho}\left[\frac{1}{2}\nabla B^2 - \nabla\cdot(\boldsymbol{B}\boldsymbol{B})\right]$$

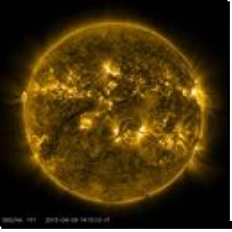
$$= -\frac{\nabla P}{\rho} - \frac{1}{\mu_0\rho}\left[\frac{1}{2}\nabla B^2 - \left\{\frac{1}{2}\nabla B^2 - \boldsymbol{B}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{B}(\nabla\cdot\boldsymbol{B})\right\}\right]$$

$$= -\frac{\nabla P}{\rho} + \frac{(\nabla\times\boldsymbol{B})\times\boldsymbol{B}}{\mu_0\rho} + \frac{\boldsymbol{B}(\nabla\cdot\boldsymbol{B})}{\mu_0\rho}$$

$= 0$ physically

$\neq 0$ numerically

- The magnetic monopole term exists when the equations conserve energy



# *SPMHD: Tensile Instability*

- Momentum equation inherently includes $\nabla.\mathbf{B} \neq 0$, thus needs to be removed (Børve, Omang & Trulsen 2001, 2004; Tricco & Price 2012)

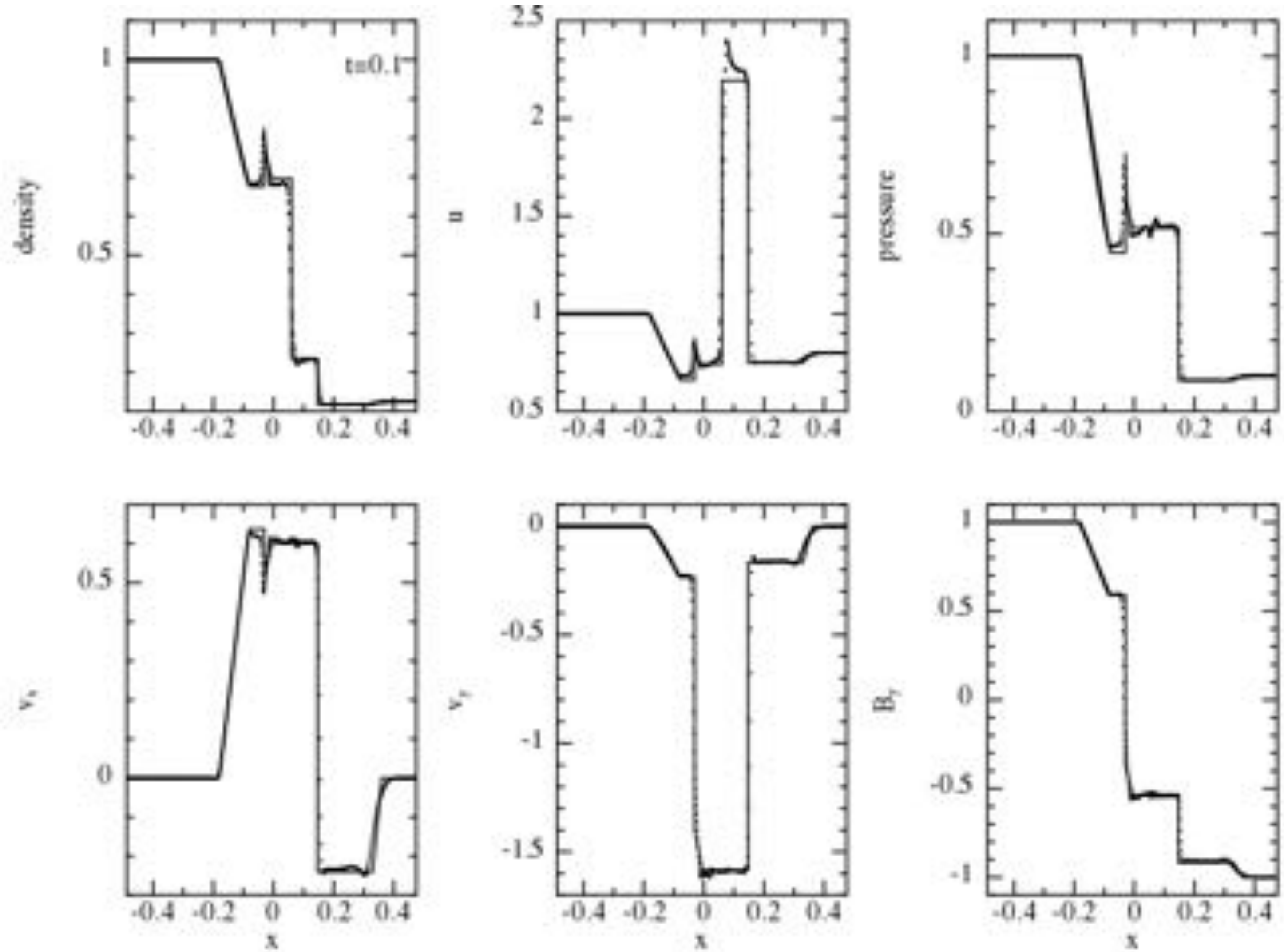
$$\frac{\mathrm{d}v_a^i}{\mathrm{d}t} = \sum_b m_b \left[ \frac{S_a^{ij}}{\Omega_a \rho_a^2} \nabla_a^j W_{ab}(h_a) + \frac{S_b^{ij}}{\Omega_b \rho_b^2} \nabla_a^j W_{ab}(h_b) \right] - f_a B_a^i \sum_b m_b \left[ \frac{B_a^j}{\Omega_a \rho_a^2} \nabla_a^j W_{ab}(h_a) + \frac{B_b^j}{\Omega_b \rho_b^2} \nabla_a^j W_{ab}(h_b) \right]$$

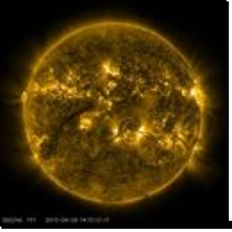
$$S_a^{ij} \equiv -\left( P_a + \frac{1}{2\mu_0} B_a^2 \right) \delta^{ij} + \frac{1}{\mu_0} B_a^i B_a^j$$

- $f_a = 1$: removes magnetic monopoles; does not conserve energy or momentum
- $f_a = 0$: conserves energy & momentum; includes unphysical magnetic monopoles
- We name $f_a > 0$ the Børve correction

# *SPMHD: Tensile Instability: Brio-Wu Shock*

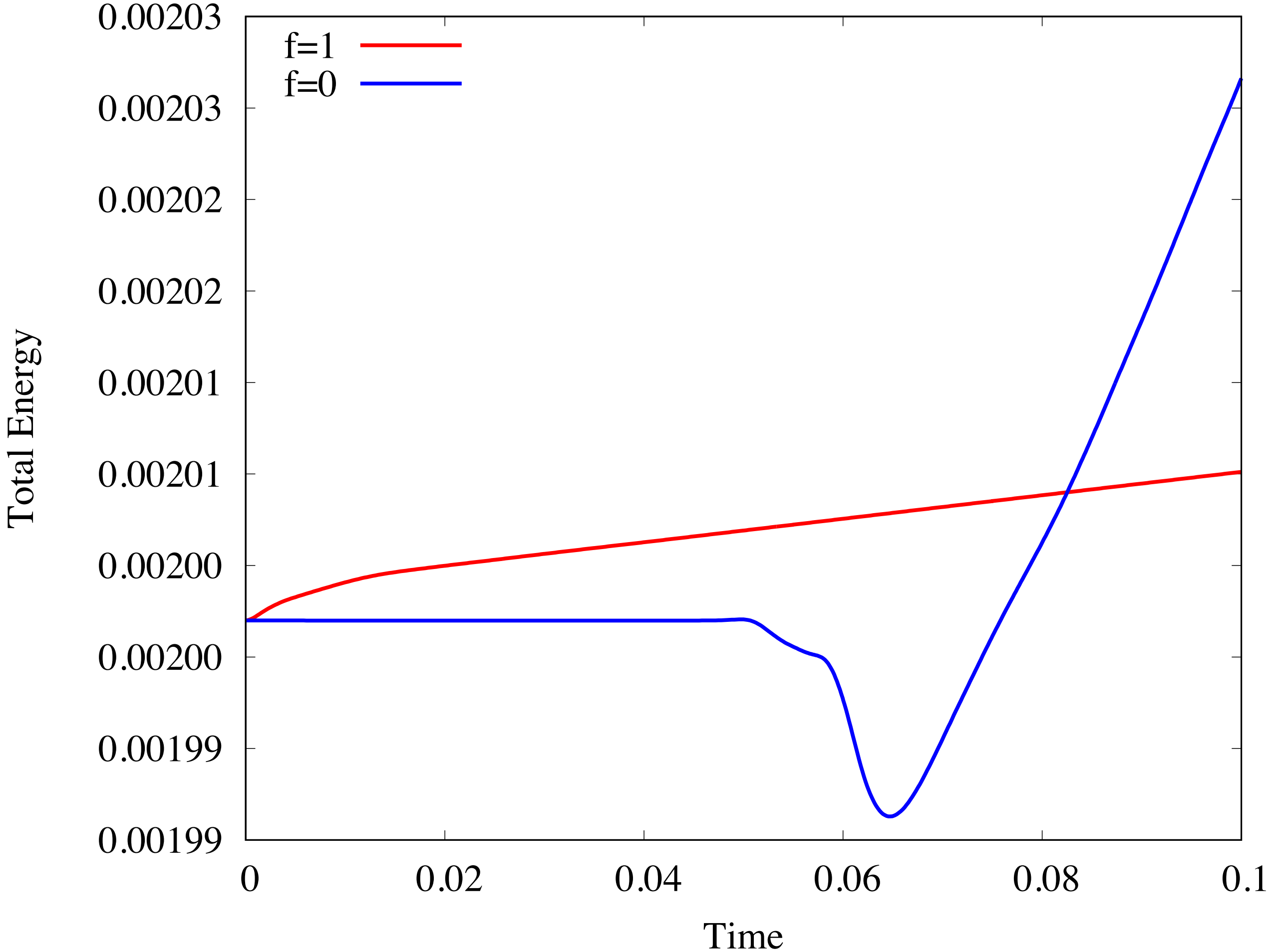
- With maximal artificial viscosity, conductivity and resistivity and Børve correction





# *SPMHD: Tensile Instability: Brio-Wu Shock*

- What is the optimal $f_a$ ?
  - $f_a = 1$: removes magnetic monopoles; does not conserve energy or momentum
  - $f_a = 0$: conserves energy & momentum; includes unphysical magnetic monopoles

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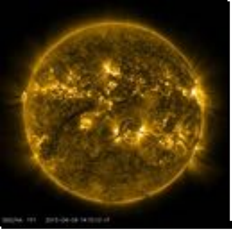
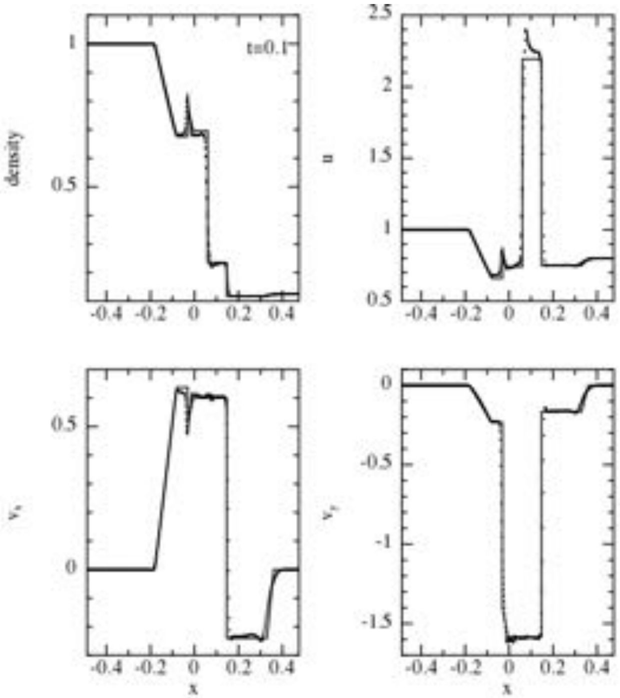


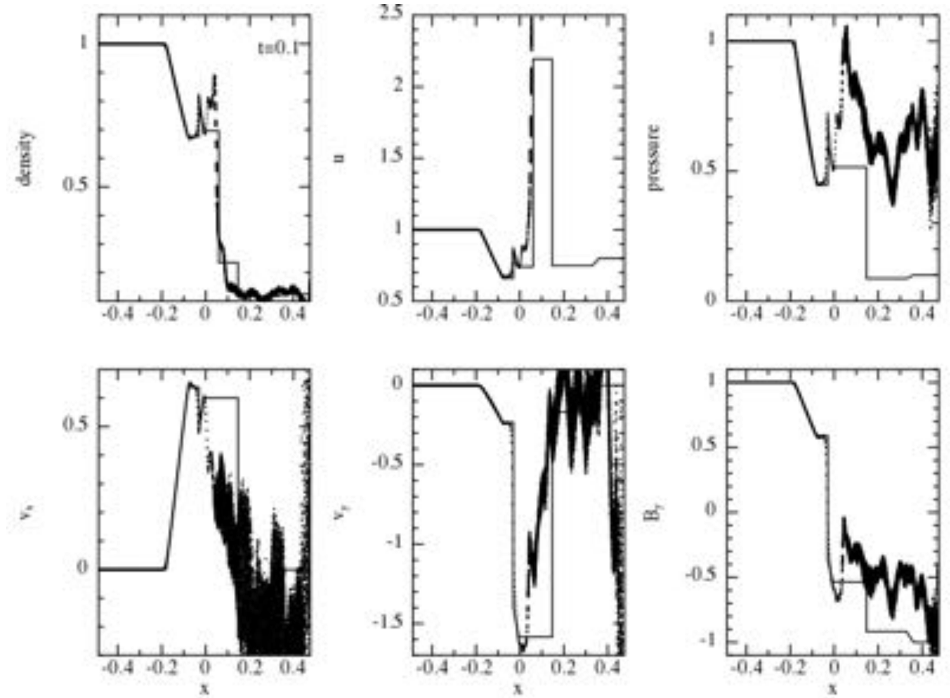
# *SPMHD: Tensile Instability: Brio-Wu Shock*

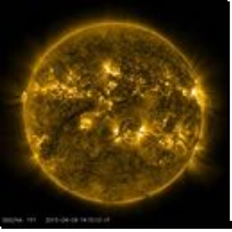
- What is the optimal $f_a$ ?
  - $f_a = 1$: removes magnetic monopoles; does not conserve energy or momentum
  - $f_a = 0$: conserves energy & momentum; includes unphysical magnetic monopoles

$f_a = 1$




$f_a = 0$

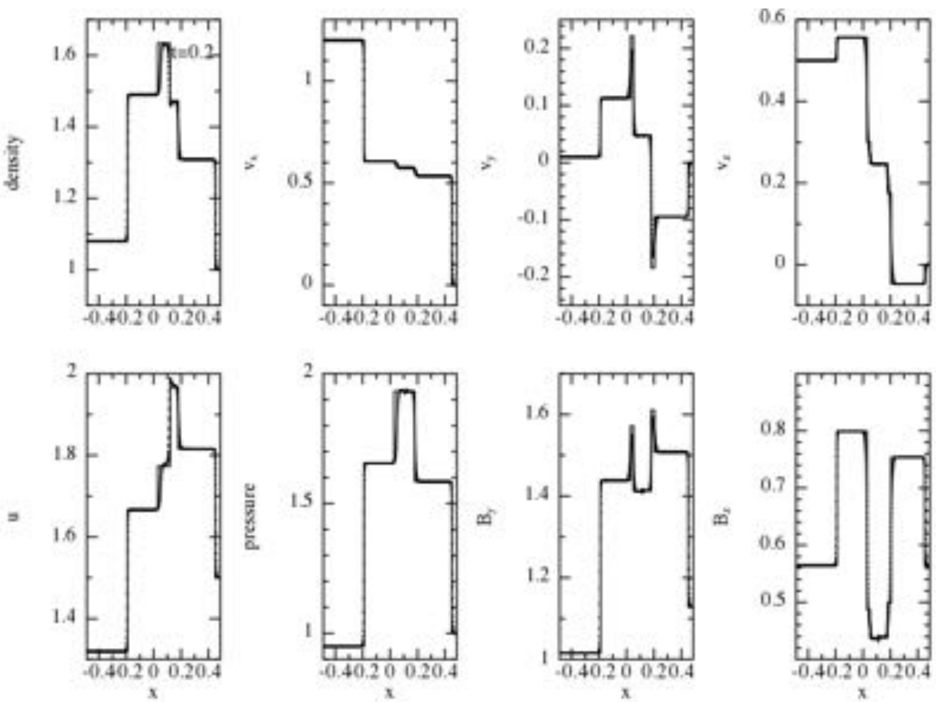




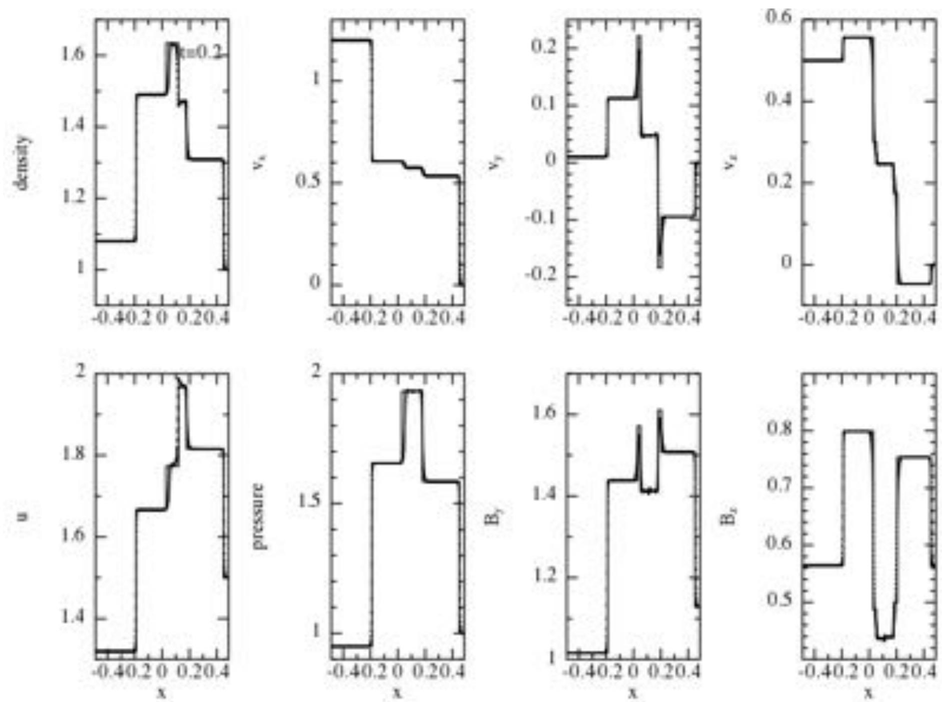
# *SPMHD: Tensile Instability: Ryu-Jones*

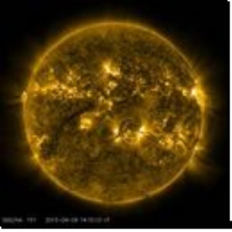
- What is the optimal $f_a$ ?
  - $f_a$ = 1: removes magnetic monopoles; does not conserve energy or momentum
  - $f_a$ = 0: conserves energy & momentum; includes unphysical magnetic monopoles

$f_a = 1$




$f_a = 0$





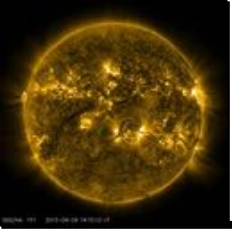
# SPMHD: Tensile Instability

- What is the optimal $f_a$ ?
  - $f_a = 1$: removes magnetic monopoles; does not conserve energy or momentum
  - $f_a = 0$: conserves energy & momentum; includes unphysical magnetic monopoles

- Recall motion in 3D without the Børve correction

$$\frac{\mathrm{d}v_a^i}{\mathrm{d}t} = \sum_b m_b \left[ \frac{S_a^{ij}}{\Omega_a \rho_a^2} \nabla_a^j W_{ab}(h_a) + \frac{S_b^{ij}}{\Omega_b \rho_b^2} \nabla_a^j W_{ab}(h_b) \right]$$

$$S_a^{ij} \equiv -\left( P_a + \frac{1}{2\mu_0} B_a^2 \right) \delta^{ij} + \frac{1}{\mu_0} B_a^i B_a^j$$



# SPMHD: Tensile Instability

- What is the optimal $f_a$ ?
  - $f_a = 1$: removes magnetic monopoles; does not conserve energy or momentum
  - $f_a = 0$: conserves energy & momentum; includes unphysical magnetic monopoles

- Recall motion in 3D without the Børve correction

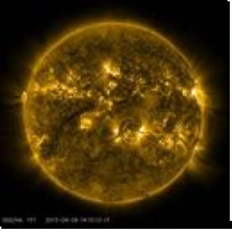
$$\frac{\mathrm{d}v_a^i}{\mathrm{d}t} = \sum_b m_b \left[ \frac{S_a^{ij}}{\Omega_a \rho_a^2} \nabla_a^j W_{ab}(h_a) + \frac{S_b^{ij}}{\Omega_b \rho_b^2} \nabla_a^j W_{ab}(h_b) \right]$$

$$S_a^{ij} \equiv -\left( P_a + \frac{1}{2\mu_0} B_a^2 \right) \delta^{ij} + \frac{1}{\mu_0} B_a^i B_a^j$$

- Consider motion in a 1-D calculation:

$$\frac{\mathrm{d}v_a^x}{\mathrm{d}t} = -\sum_b m_b \left[ \frac{P_a - \frac{1}{2\mu_0} B_{a,x}^2}{\Omega_a \rho_a^2} + \frac{P_b - \frac{1}{2\mu_0} B_{b,x}^2}{\Omega_b \rho_b^2} \right] \frac{\mathrm{d}W_{ab}}{\mathrm{d}x}$$

- If $P - \frac{1}{2\mu_0} B_x^2 < 0$ , then $\frac{\mathrm{d}v_a^x}{\mathrm{d}t} > 0$ , therefore the force is attractive, which leads to the Tensile instability



# SPMHD: Tensile Instability

- What is the optimal $f_a$ ?
  - $f_a = 1$: removes magnetic monopoles; does not conserve energy or momentum
  - $f_a = 0$: conserves energy & momentum; includes unphysical magnetic monopoles

- Consider plasma $\beta$:

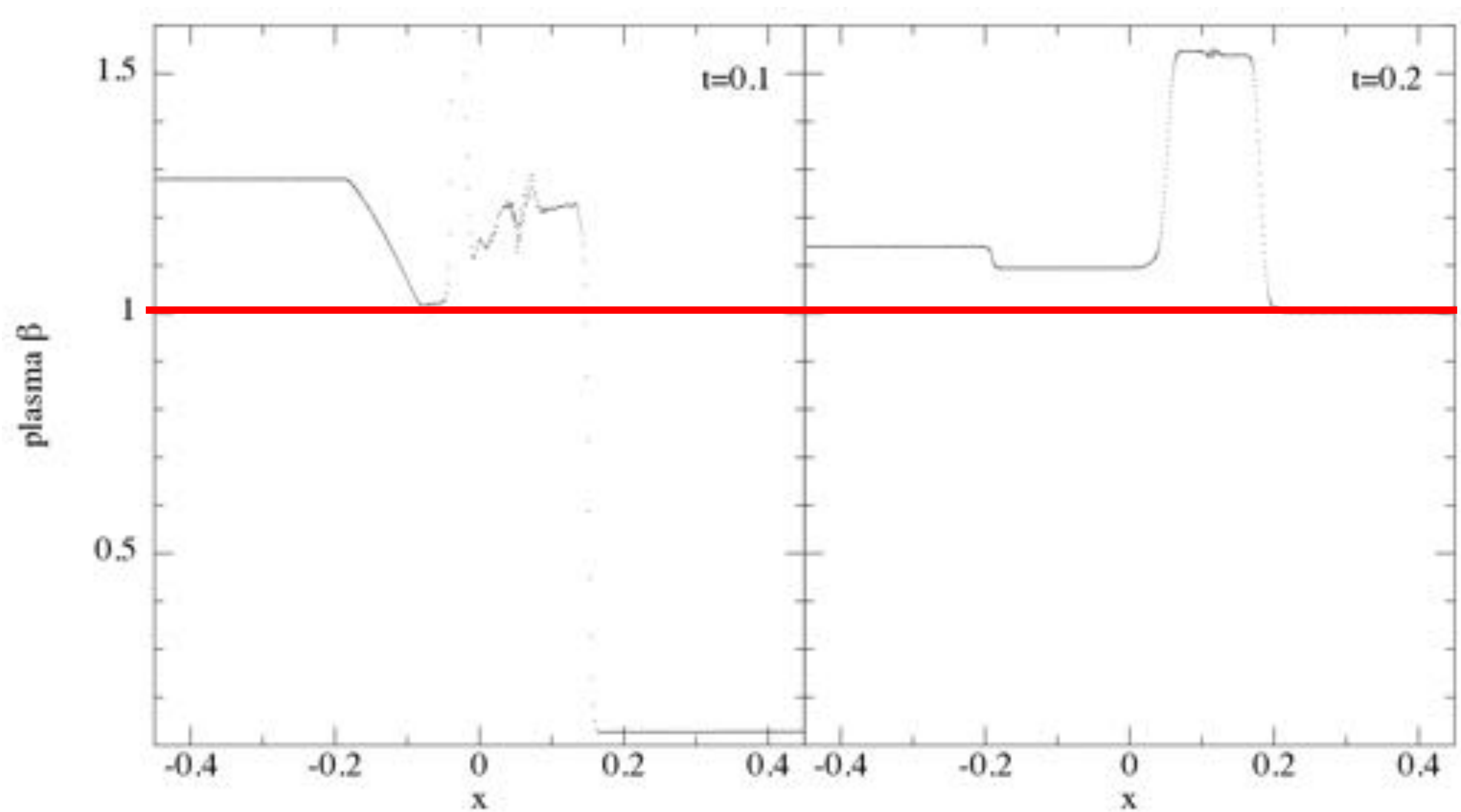
$$\beta = \frac{P_{\text{gas}}}{P_{\text{mag}}} = \frac{2\mu_0 P_{\text{gas}}}{B^2}$$

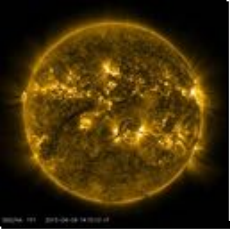
  - $\beta >> 1$: Evolution is dominated by gas pressure
  - $\beta << 1$: Evolution is dominated by magnetic pressure & leads to Tensile instability

# SPMHD: Tensile Instability: Børve correction

- What is the optimal $f_a$ ?
  - $f_a = 1$: removes magnetic monopoles; does not conserve energy or momentum
  - $f_a = 0$: conserves energy & momentum; includes unphysical magnetic monopoles

Brio-Wu (unstable with $f_a = 0$) Ryu-Jones (stable with $f_a = 0$)



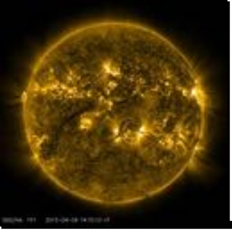


# *SPMHD: Tensile Instability: Børve correction*

- What is the optimal $f_a$ ?
  - $f_a = 1$ (Børve, Omang & Trulsen 2001; Tricco & Price 2012)
  - $0 < f_a < ½$ (Børve, Omang & Trulsen 2004)

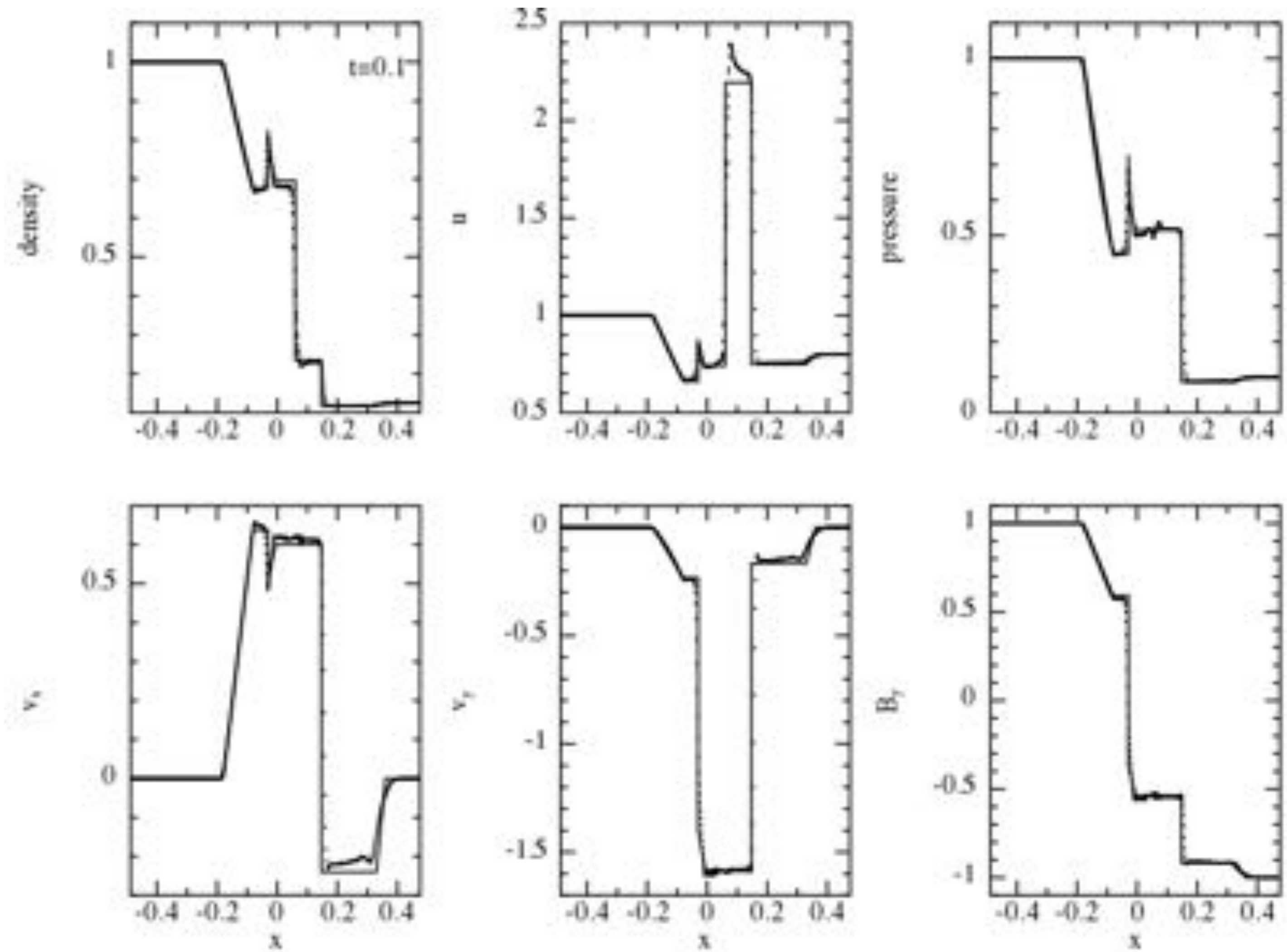
- If unstable only for magnetically dominated regions (i.e. $\beta < 1$) , then only subtract there:

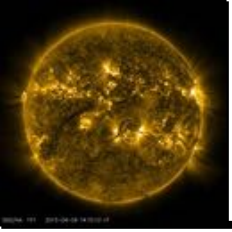
$$f_a = \begin{cases} 1; & \beta_a \leq 1 \\ 2 - \beta; & 1 < \beta_a \leq 2 \\ 0; & \beta_a > 2 \end{cases}$$



# *SPMHD: Tensile Instability: Børve correction*

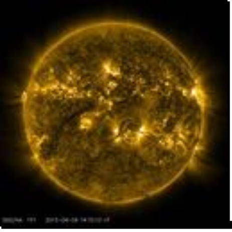
- $f_a \equiv f_a(\beta)$





# ***SPMHD: Divergence Cleaning***

- As previously shown, $\nabla \cdot \boldsymbol{B} = 0$ is not guaranteed by the SPMHD equations
  - Option 1: Ignore
    - Monitor $\frac{h\nabla \cdot \boldsymbol{B}}{|B|}$ and *hope* it remains small



# SPMHD: Divergence Cleaning

- As previously shown, $\nabla \cdot \boldsymbol{B} = 0$ is not guaranteed by the SPMHD equations
  - Option 2: Clean
    - e.g. constrained hyperbolic/parabolic divergence cleaning (Tricco, Price & Bate, 2016 using Dedner et al 2002 and Price & Monaghan 2005)
    - Introduce a (non-physical) scalar field, $\psi$, with energy

$$e_{\psi} = \frac{\psi^2}{2\mu_0 \rho c_{\mathrm{h}}^2}.$$

    - Continuum equations

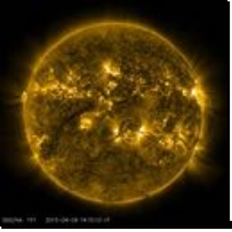
$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = (\mathbf{B} \cdot \nabla)\mathbf{v} - \mathbf{B}(\nabla \cdot \mathbf{v}) - \nabla \psi,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\psi}{c_{\mathrm{h}}}\right) = -c_{\mathrm{h}}(\nabla \cdot \mathbf{B}) - \frac{1}{\tau}\left(\frac{\psi}{c_{\mathrm{h}}}\right) - \frac{1}{2}\left(\frac{\psi}{c_{\mathrm{h}}}\right)(\nabla \cdot \mathbf{v}).$$

    where $c_h$ is the characteristic or wave-cleaning speed
    - Discrete equations

$$\left(\frac{\mathrm{d}\mathbf{B}_a}{\mathrm{d}t}\right)_{\psi} = -\rho_a \sum_b m_b \left[\frac{\psi_a}{\Omega_a \rho_a^2} \nabla_a W_{ab}(h_a) + \frac{\psi_b}{\Omega_b \rho_b^2} \nabla_a W_{ab}(h_b)\right],$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\psi}{c_{\mathrm{h}}}\right)_a = \frac{c_{\mathrm{h},a}}{\Omega_a \rho_a} \sum_b m_b (\mathbf{B}_a - \mathbf{B}_b) \cdot \nabla_a W_{ab}(h_a) - \frac{1}{\tau}\left(\frac{\psi}{c_{\mathrm{h}}}\right)_a + \frac{1}{2}\left(\frac{\psi}{c_{\mathrm{h}}}\right)_a \sum_b m_b (\mathbf{v}_a - \mathbf{v}_b) \cdot \nabla_a W_{ab}(h_a).$$



# SPMHD: Divergence Cleaning

- As previously shown, $\nabla \cdot \boldsymbol{B} = 0$ is not guaranteed by the SPMHD equations
  - Option 2: Clean
    - e.g. constrained hyperbolic/parabolic divergence cleaning
      (Tricco, Price & Bate, 2016 using Dedner et al 2002 and Price & Monaghan 2005)
    - Introduce a (non-physical) scalar field, $\psi$, with energy

$$e_\psi = \frac{\psi^2}{2\mu_0 \rho c_\mathrm{h}^2}.$$

- Hyperbolic Transport
- Parabolic damping
- Compression / rarefaction

    - Continuum equations

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = (\mathbf{B}\cdot\nabla)\mathbf{v} - \mathbf{B}(\nabla\cdot\mathbf{v}) - \nabla\psi,$$

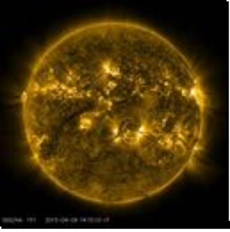
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\psi}{c_\mathrm{h}}\right) = -c_\mathrm{h}(\nabla\cdot\mathbf{B}) - \frac{1}{\tau}\left(\frac{\psi}{c_\mathrm{h}}\right) - \frac{1}{2}\left(\frac{\psi}{c_\mathrm{h}}\right)(\nabla\cdot\mathbf{v}),$$

where $c_h$ is the characteristic or wave-cleaning speed

    - Discrete equations

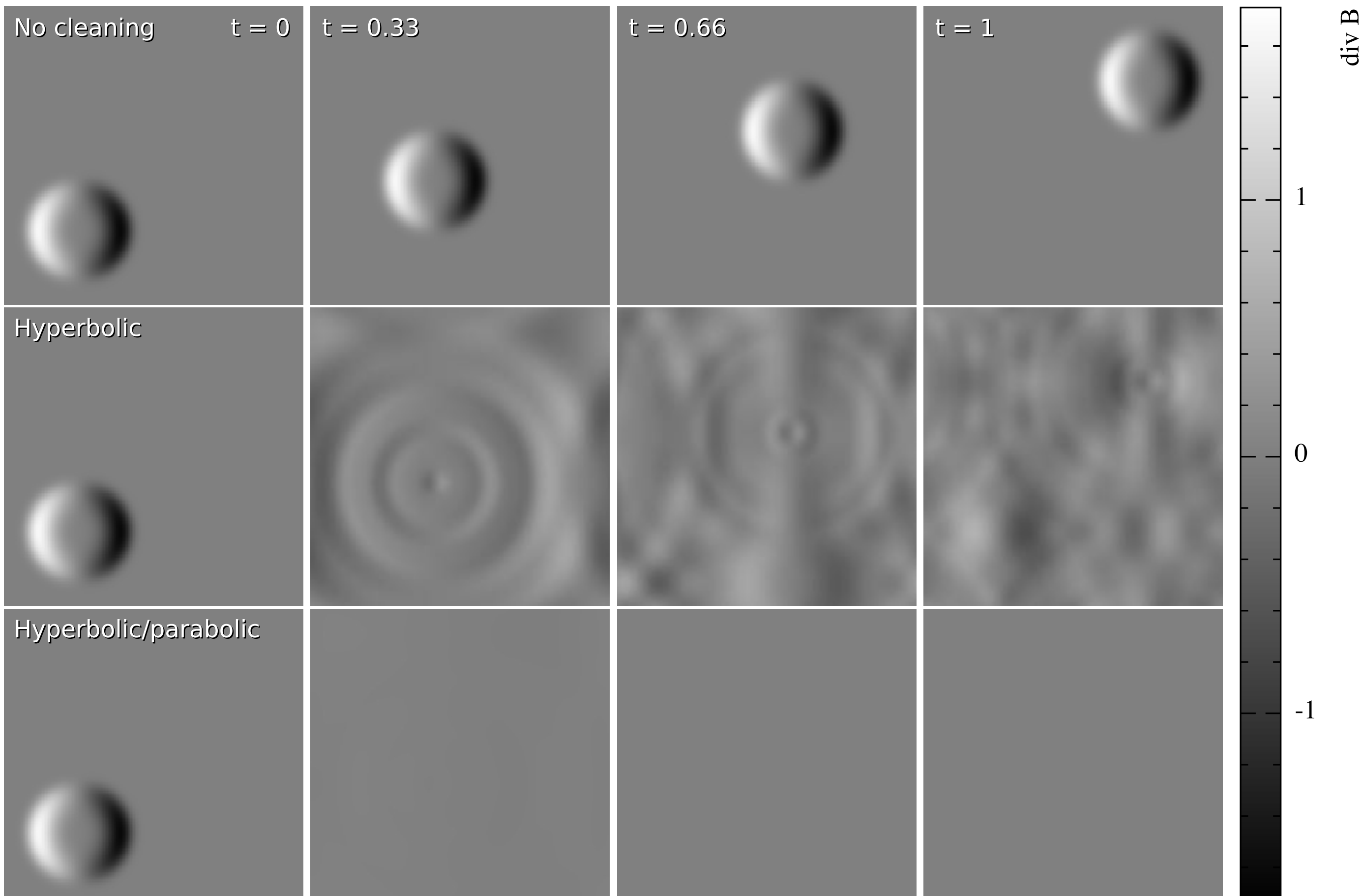
$$\left(\frac{\mathrm{d}\mathbf{B}_a}{\mathrm{d}t}\right)_\psi = -\rho_a \sum_b m_b \left[\frac{\psi_a}{\Omega_a \rho_a^2}\nabla_a W_{ab}(h_a) + \frac{\psi_b}{\Omega_b \rho_b^2}\nabla_a W_{ab}(h_b)\right],$$

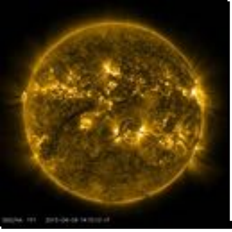
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\psi}{c_\mathrm{h}}\right)_a = \frac{c_{\mathrm{h},a}}{\Omega_a \rho_a}\sum_b m_b(\mathbf{B}_a - \mathbf{B}_b)\cdot\nabla_a W_{ab}(h_a) - \frac{1}{\tau}\left(\frac{\psi}{c_\mathrm{h}}\right)_a + \frac{1}{2}\left(\frac{\psi}{c_\mathrm{h}}\right)_a \sum_b m_b(\mathbf{v}_a - \mathbf{v}_b)\cdot\nabla_a W_{ab}(h_a).$$



# SPMHD: Divergence Cleaning

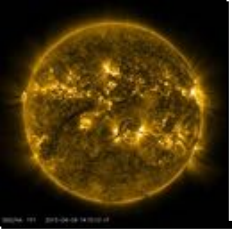
- As previously shown, $\nabla \cdot \boldsymbol{B} = 0$ is not guaranteed by the SPMHD equations
  - Option 2: Clean
    - e.g. constrained hyperbolic/parabolic divergence cleaning (Tricco, Price & Bate, 2016); Fig 1.





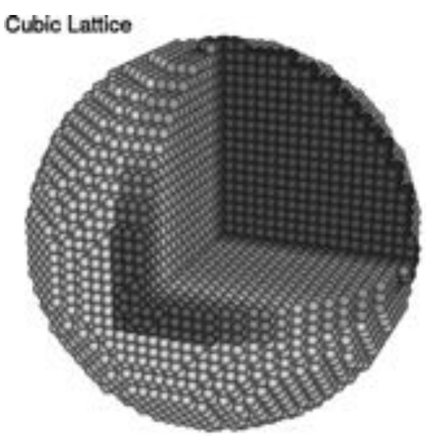
# *SPMHD: Divergence Cleaning*

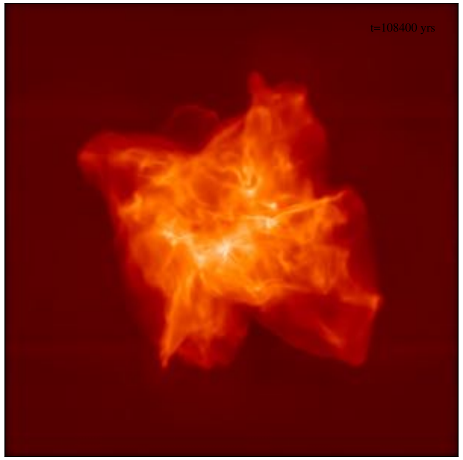
- As previously shown, $\nabla \cdot \boldsymbol{B} = 0$ is not guaranteed by the SPMHD equations
  - Option 3: Avoid (i.e. construction equations that enforce divergence-free)
    - Constrained transport used by grid codes is divergence-free
    - Constrained transport cannot be applied to SPMHD since it required computation of surface rather than volume integrals.
    - Euler Potentials
      - Closest analogy to constrained transport in SPMHD
      - Cannot represent winding motion or prevent dynamo processes
      - Non-trivial to implement resistive terms
      - Been used (e.g. Price & Bate 2007, 2008, 2009) and since abandoned in favour of the induction equation and cleaning
    - Vector Potentials
      - Numerically unstable (Price 2010)

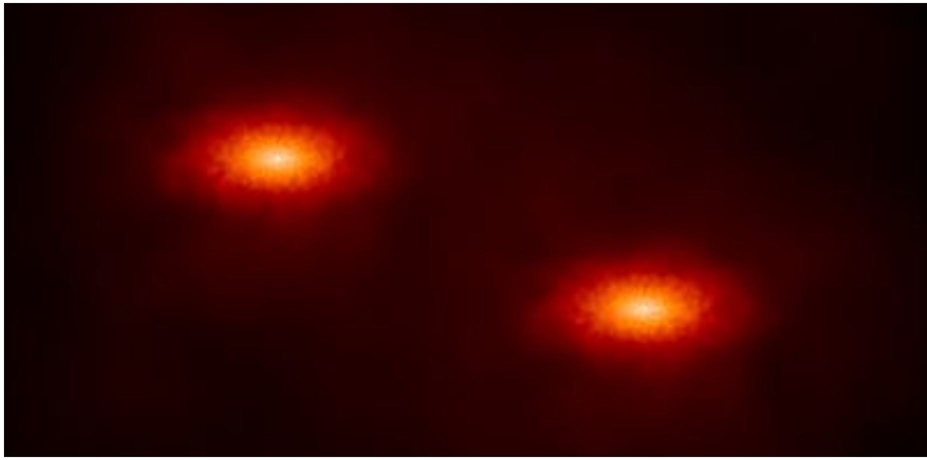


# *SPMHD: Boundary Conditions*

- Boundaries are optional in SPH
  - e.g. collapse of a sphere
  - e.g. evolution of a turbulent sphere (Bate's cluster models)
  - e.g. galaxy merger simulation (Wurster & Thacker 2013a,b)

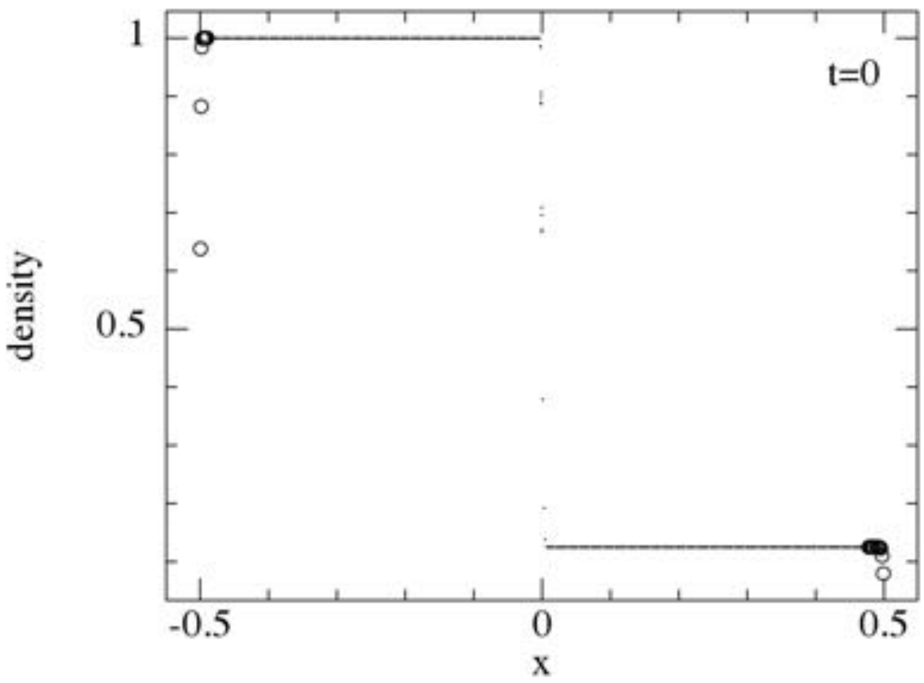


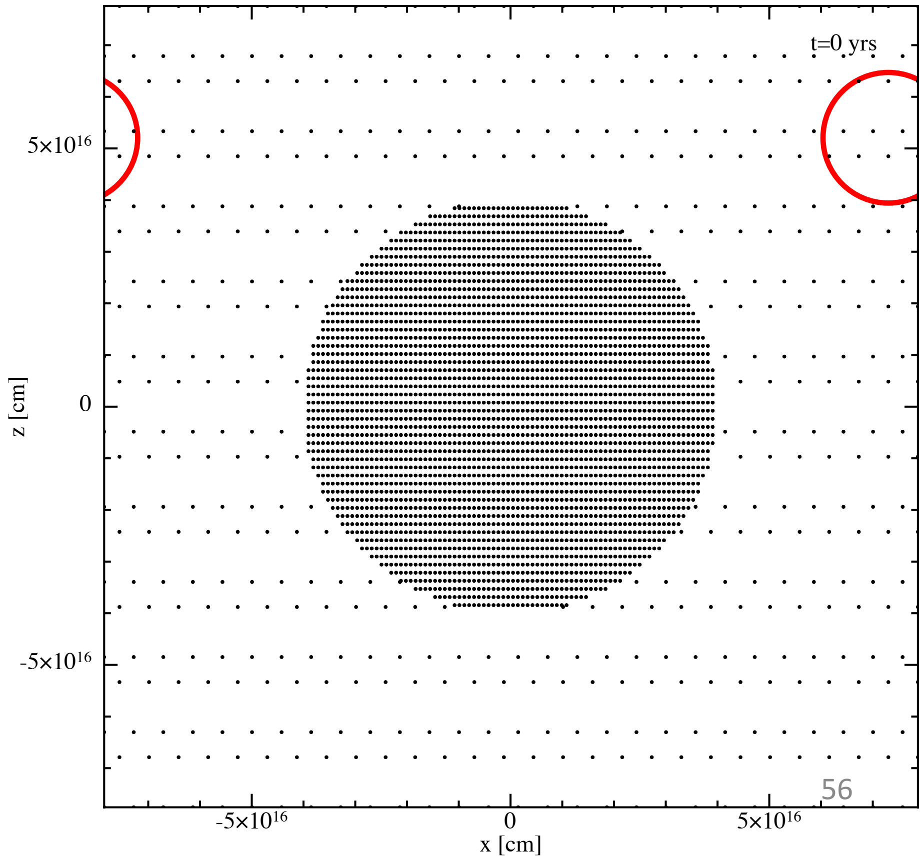



# *SPMHD: Boundary Conditions*

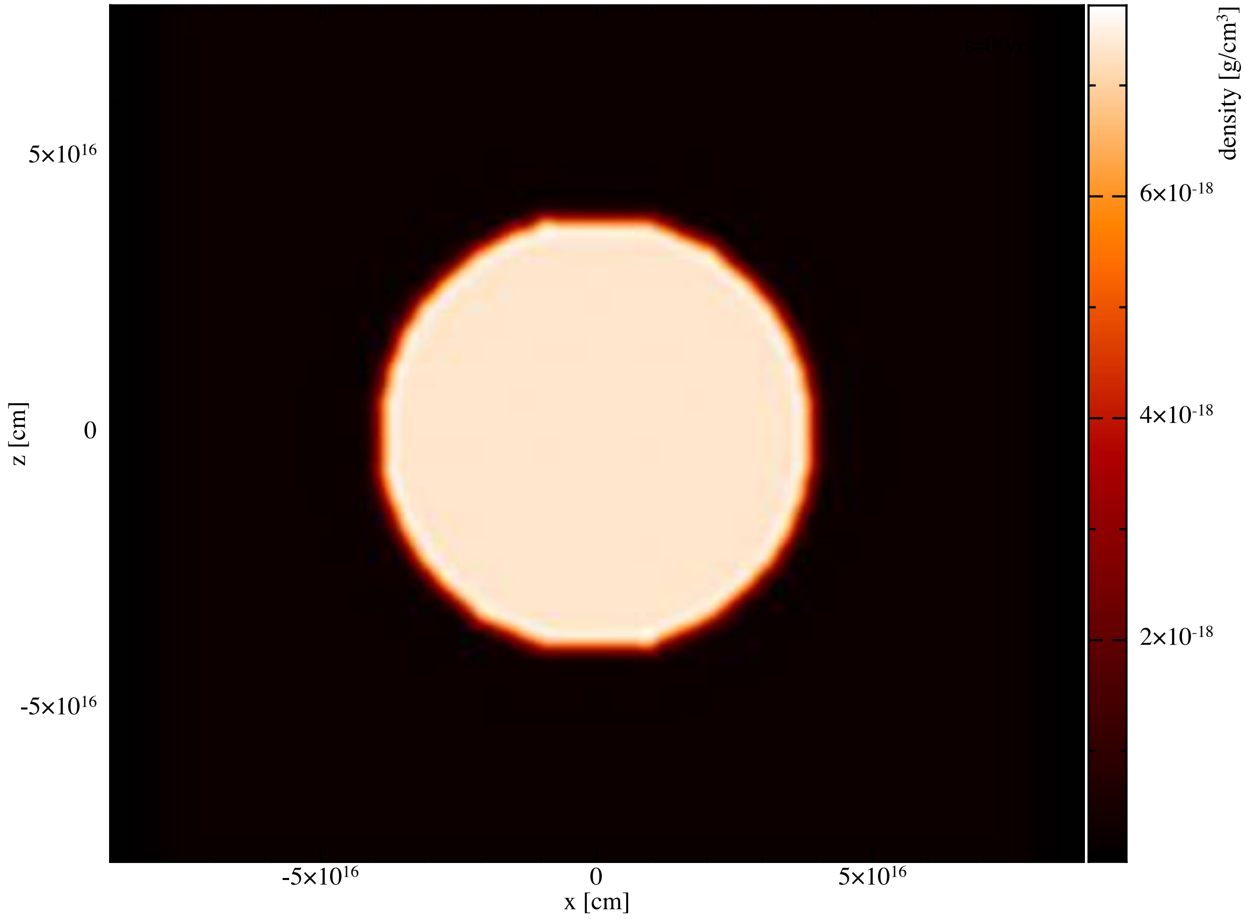
- Boundaries in SPH
  - Fixed
  - Periodic
- For Sod shock tube, periodic in *y* & *z* and fixed in *x*
- In the image, dot are active particles, circles are boundary particles



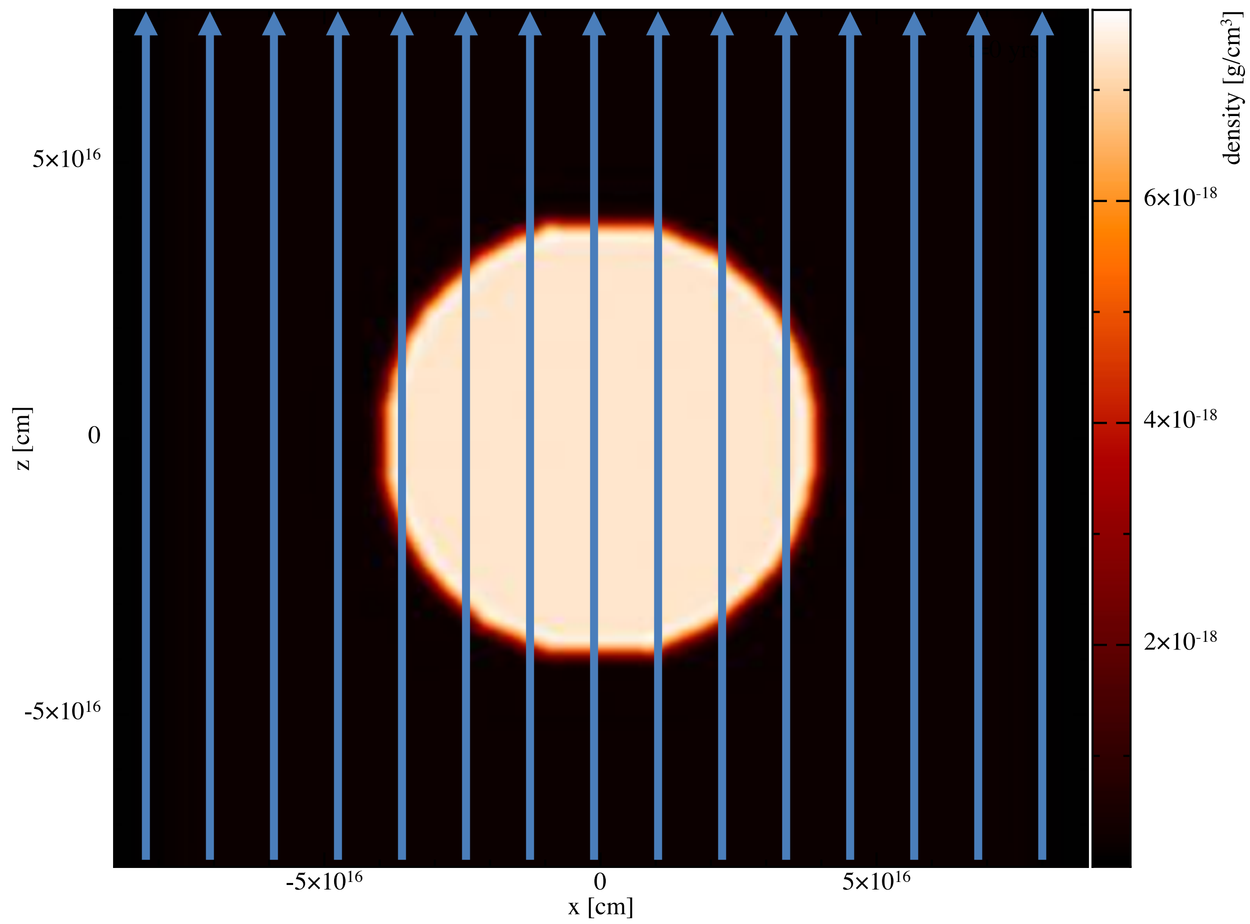
# *SPMHD: Boundary Conditions*

- How do we treat boundaries for the gravitational collapse of a magnetised sphere?

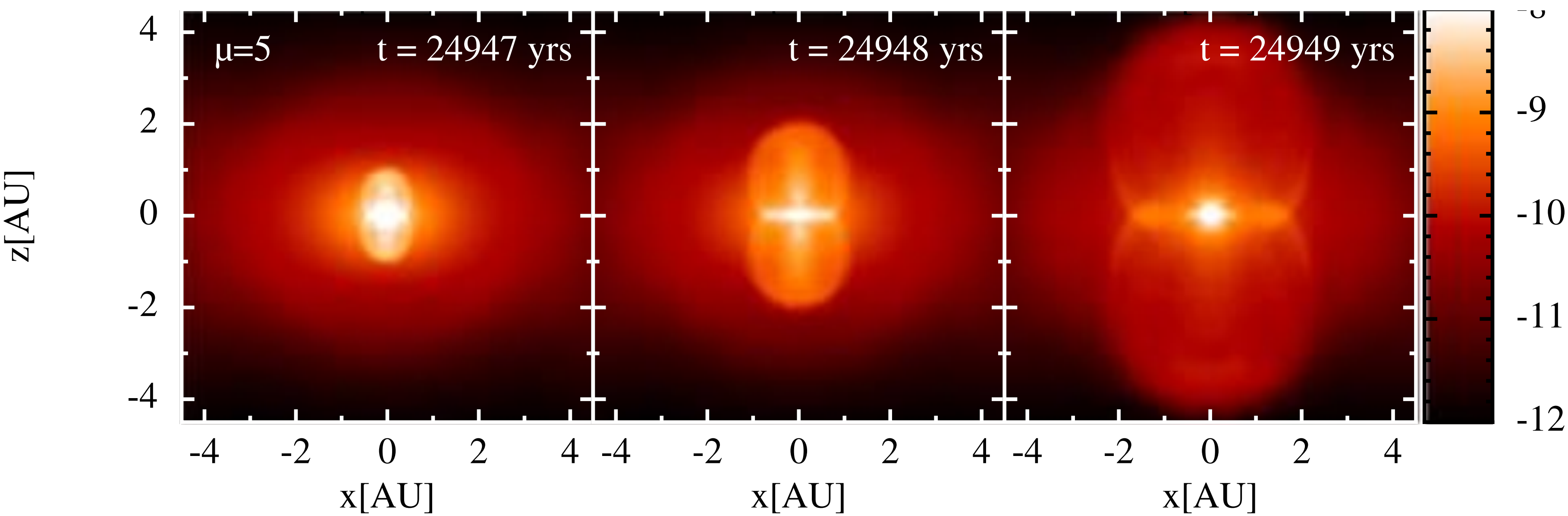


# SPMHD: Boundary Conditions

- Embed sphere in low density medium (e.g. with density ratio 30:1)
- Thread magnetic field throughout the entire domain
- Use periodic boundaries at the edge of the box



# *SPMHD: Example: Star formation*

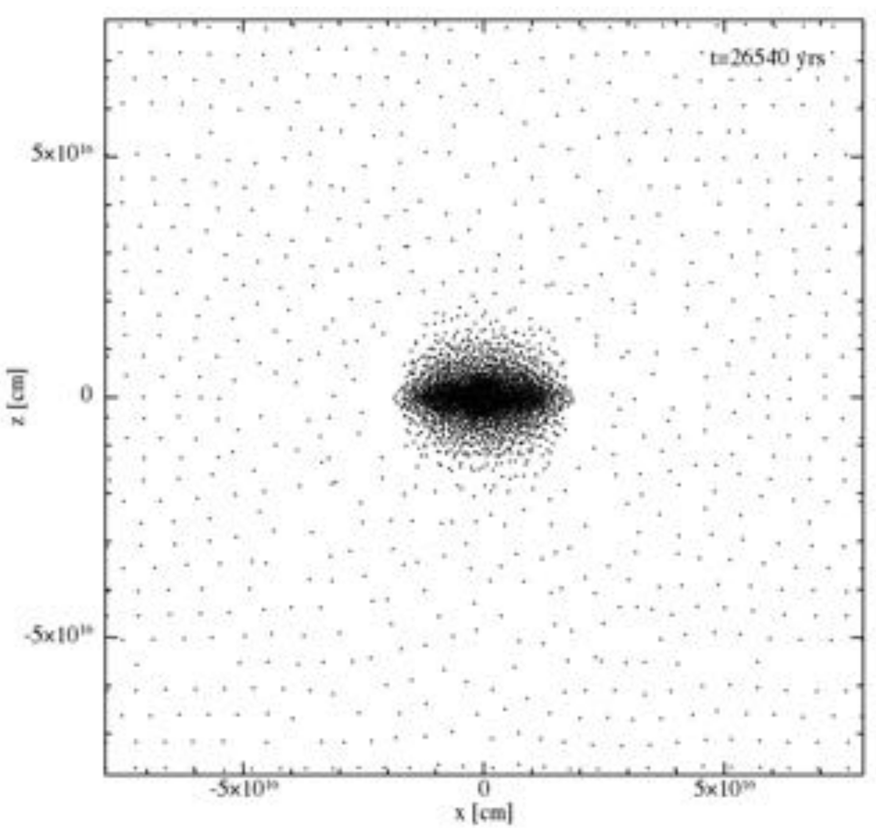


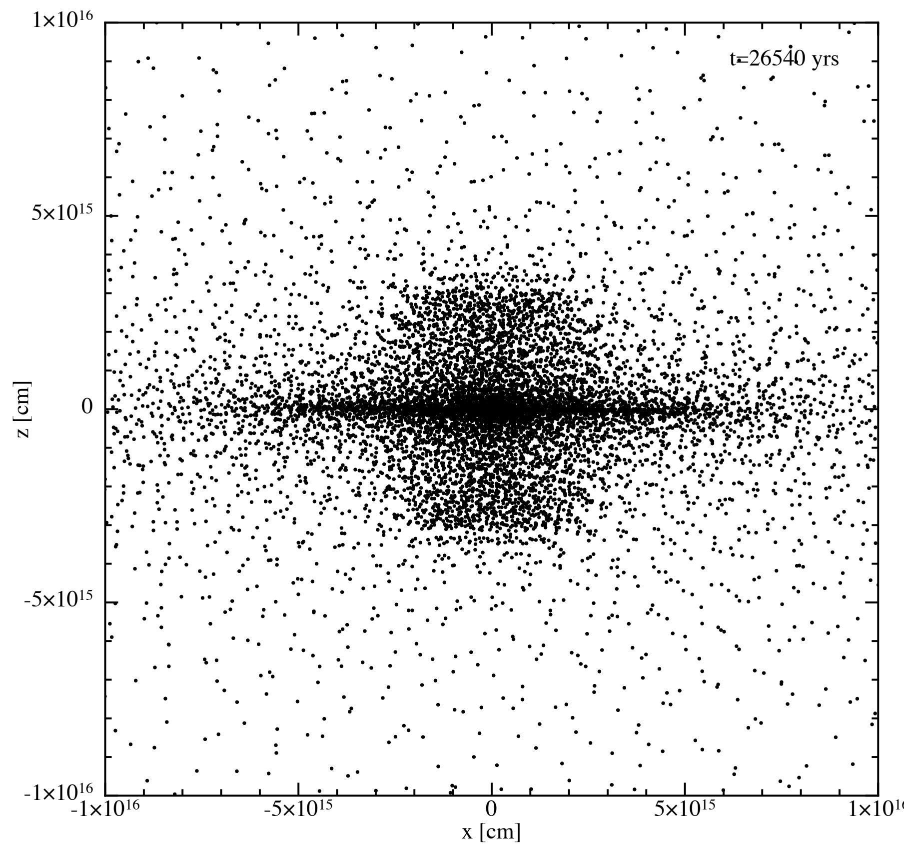

E.g. Bate, Tricco & Price (2014)

Video: https://www.astro.ex.ac.uk/people/mbate/Animations/BateTriccoPrice2013_MF05.mov

# *SPMHD: Example: Star formation*

- Cross section of particle positions



# *Running Phantom*

- Install Phantom in the home directory

- In your run-directory, call

**phantom/scripts/writemake.sh SETUP > Makefile**

- where SETUP = shock for the Sod shock tube test, or SETUP = mhdshock for MHD shocks

- Compile Phantom via

**make SYSTEM=gfortran; make setup SYSTEM=gfortran**

- where gfortran can be replaced with other systems, e.g., ifort

- Run phantom setup via

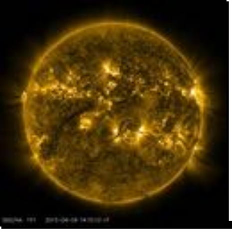
**./phantomsetup INFILE**

- where INFILE is the run name (e.g.) sod

- Modify INFILE.in as required

- Run phantom via

**./phantom INFILE.in**



# *Conclusions*

- Particles of fixed mass are used to represent the fluid
- Properties are calculated at a particle's position using a smoothing kernel and its neighbours
- Smoothing length is adaptive
- Smoothed particle magnetohydrodynamics requires
  - artificial resistivity for stability
  - subtraction of magnetic monopole from equation of motion (tensile instability)
  - divergence cleaning to remove magnetic monopoles
  - boundaries
- This presentation is available at: http://www.astro.ex.ac.uk/people/wurster/files/spmhd.pdf
- Contact info: j.wurster [at] exeter.ac.uk