Cosmology applies physics to the universe as a whole, describing it’s origin, nature evolution and ultimate fate.

While these questions have been the topics of religion and philosophy since the beginning of civilisation, Modern Cosmology as a science began in 1917 when Einstein used the newly completed theory of GR to describe the universe.

With GR, a cosmological model tries to describe the large scale features of the universe. These models provide the basis for modern Relativistic cosmology.

Currently, cosmology is undergoing a golden age, with the most startling and important features only discovered and described within the last decade. Compared to only 15 years ago, we now know the universe is 13.7 billion years old, and is currently comprised mostly of dark energy.
Three underlying principles will guide our cosmological models

1) General Relativity alone is sufficient to describe the large-scale features of the universe

2) The cosmological principle

3) Weyl’s postulate.

(1) The applicability of GR and gravity to describe large-scale features is “obvious” upon close inspection. Given the current size of the universe, two of the fundamental forces of nature (strong & weak force) operate on much to short of a distance to be useful on a super-galactic scale. Plus the electromagnetic force is almost always balanced, thus it’s effects are canceled at any large, meaningful scale. This leaves gravity as the lone force which can dictate how the universe itself evolves. Moreover as gravity is geometry, GR addresses the basic question about the geometrical shape of the universe.
2) The cosmological principle

Observations of distant galaxies and quasars have shown the universe to be approximately homogeneous and isotropic on spacial scales larger than a few hundred million light years. The simplest cosmological models enforce these symmetries exactly as a first approximation.

**Cosmological Principle** - At any given time, and at a sufficiently large scale, the universe is **Homogeneous** and **Isotropic**.

With these assumptions the matter in galaxies and radiation are approximated by smooth density distributions, that are exactly uniform in space. Similarly, the geometry of spacetime incorporates the homogeneity and isotropy of space exactly.

A homogenous, isotropic spacetime is one for which the geometry is spherically symmetric about any one point in space (isotropic) and the same at any one point in space as any other point (homogeneous).
The homogeneity and isotropy are symmetries of space not spacetime. Spacetime can still be curved even if the 3D space is flat.

At first, and to our every-day experiences, the universe is anything but homogeneous or isotropic. Only on the largest scales is this true. Let's see what scales we are talking about...
Large Scale Structure in the Local Universe

Legend: image shows 2MASS galaxies color coded by redshift (Jarrett 2004); familiar galaxy clusters/supercusters are labeled (numbers in parenthesis represent redshift). Graphic created by T. Jarrett (IPAC/Caltech)
Cosmic Web - dark voids and a spidery mesh of galactic superclusters. Characteristic scale is \( \sim 100 \) Mlyr. These are believed to be the largest structures in the Universe.
At larger scales beyond the cosmic web (billions of Lyr), the Universe is approximately isotropic and homogeneous.

It has taken light a good faction of the age of the whole universe (5-10%) for light to cross these homogeneous/isotropic scales.
The simplest example of a homogenous isotropic cosmological geometry is described by the line element

\[ ds^2 = c^2 dt^2 - a^2(t)[dx^2 + dy^2 + dz^2] \]

Which is the flat Robertson-Walker metric. \( a(t) \) is a function of the coordinate time \( t \) and is the **scale factor**.

The metric is homogeneous and isotropic because it can be divided into the time component, and a homogeneous isotropic spacial component.

\[ ds^2 = c^2 dt^2 - dS^2 \]

at any instant in time, \( t \)

space is described by the line element

\[ dS^2 = a^2(t)[dx^2 + dy^2 + dz^2] \]

We now know the Universe is non-static, and currently accelerating. The scale factor describes the average large-scale motions called the **Hubble flow**, which is isotropic and can be described by a single rate at any given time.
Weyl’s postulate is an assumption about the nature of the Hubble flow, and recognises there exists a certain privileged class of observers who move with the Hubble flow, giving them a simple view of the universe.

One usually pictures such privileged observes as galaxies or galaxy clusters who then view every other privileged observer (galaxy) as moving away from them in a uniformly expanding universe.
Individual galaxies can have their own peculiar motion, but the large scale structures evolve in a sufficiently orderly way such that a single meaningful **cosmic time** can be associated with every event. Essentially, these fundamental observers are at rest with respect to the universe, and so measure and will agree upon their proper time, from say the time since the big bang.

Useful in cosmology, **co-moving coordinates** expand and contract with the evolution of the universe, so at any time $t$ new coordinates.

\[
\begin{align*}
X &= a(t)x \\
Y &= a(t)y \\
Z &= a(t)z \\
dS^2 &= X^2 + Y^2 + Z^2
\end{align*}
\]

The geometry $dS^2$ at each constant time $t$ describes a flat homogeneous isotropic space.
Robertson-Walker Metric in the 1930s, it was shown that a just single spacetime metric underlies all relativistic models that are homogeneous and isotropic.

In its most basic form, a fundamental observer will naturally describe the line element of the form

\[ ds^2 = c^2 dt^2 - g_{ij} dx^i dx^j \]

Where \( t \) is the cosmic time, \( x^i \) are co-moving coordinates, and \( g_{ij} \) are functions of \( x^i \) and \( t \).

With a scaling factor which preserves isotropy and homogeneity, we can write the metric coefficients as:

\[ h_{ij} = g_{ij} / a^2(t) \]

The curvature of the universe must also be fixed and the same everywhere.
Robertson-Walker Metric

The metric that describes a 3D space of constant curvature is

\[ ds^2 = c^2 dt^2 - a^2(t) \left[ \frac{d\bar{r}^2}{1 - K \bar{r}^2} + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2 \theta d\phi^2 \right] \]

For such a metric, the Ricci curvature scalar is

\[ R = -6K \]

and it is said that space has the curvature \( K \).

The scaling factor \( a(t) \) rescales this curvature for a given time \( t \), producing a curvature \( K/a^2(t) \)

The total effect is then like blowing up a balloon, when the radius goes up by a factor of 2, the curvature decreases by a factor of 4.
What is used in practice instead of the curvature $K$ itself is the curvature parameter $k$. Which tells whether the curvature is positive, negative, or zero, so the curvature parameter can have the values of $(-1, 0, +1)$.

A coordinate transformation of $r = \bar{r} |K|^2$ for $k \neq 0$

$$r = \bar{r}$$

for $k = 0$


gives the Robertson-Walker Metric

\[ ds^2 = c^2 dt^2 - R^2(t) \left[ \frac{dr^2}{(1 - kr^2)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \]

Imposing isotropic and homogeneity for a 3D space results in their only being three general possible geometries for space, corresponding to the curvature parameter.
Flat empty space \( k=0 \) \( R(t) = \text{constant} \)

In this case, space is flat and unchanging. The RW metric reduces to the Minkowski metric, \( R(t) = c \) which gets absorbed into the co-moving coord

\[
ds^2 = c^2 dt^2 - R^2(t) \left[ \frac{dr^2}{(1 - kr^2)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]
\]

\[
ds^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 = \eta_{\alpha\beta}
\]

Flat space means the Ricci tensor and Riemann tensor are zero, thus the stress-energy tensor is zero as well.

\[
R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\frac{8\pi G}{c^4} T_{\alpha\beta} = 0
\]

This describes an empty universe with no gravity, thus no matter or radiation. This solution is trivial, but interesting to see the RW reduces to the Minkowski metric.
Flat non-empty space  \( k=0 \)  \( R(t) \neq 0 \)

In this case, space is flat but expanding or contracting. The RW metric reduces to a form close the Minkowski metric, but with the added scale factor

\[
 ds^2 = c^2 dt^2 - R^2(t) \left[ \frac{dr^2}{(1 - kr^2)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]
\]

\[
 ds^2 = c^2 dt^2 - R^2(t) \left[ dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]
\]

In this Universe, parallel lines remain parallel, and a triangle is still 180 degrees as the 3D spacial hypersurface still has euclidean geometry, but the 4D spacetime is still curved. The changing scale factor causes the distances between co-moving coordinates to change, which prevents the Riemann curvature tensor from being zero.

So there is gravity (matter & radiation) as 4D space is curved, but 3D space is flat. To the best of our knowledge this is the universe we live in now. From WMAP  \( K=0 \) to about 1%
Closed FRW  \( k = +1 \)  \( R(t) \neq 0 \)

In this case, space has a positive curvature, described by a three sphere. The space is **closed**, with a finite volume, but has no boundary. It is the 3-D analog of the surface of a sphere, which has a finite area but no boundary. We can compute the 3-volume of a spacial slice of the RW metric model by integrating over the coordinates

\[
ds^2 = c^2 dt^2 - R^2(t) \left[ \frac{dr^2}{(1 - r^2)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]
\]

\[
r = \sin \chi
\]

\[
dV = \sqrt{g_{11}g_{22}g_{33}} dx^1 dx^2 dx^3
\]

\[
V(t) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\pi} R^3(t) d\chi \sin^2 \chi \sin \theta
\]

\[
V(t) = 2\pi^2 R^3(t)
\]

As the universe gets bigger, the volume expands. In this universe, traveling on a straight line would eventually travel back to points visited before.
Open FRW \( k=-1 \) \( R(t)\neq 0 \)

In this case, space has a negative curvature, described by the geometry of a Lorentz-hyperboloid. Spacial slices of this model have infinite volume and are therefore called open.

\[
ds^2 = c^2 dt^2 - R^2(t) \left[ \frac{dr^2}{1 + r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]
\]

The analogy is with the “saddle” surface geometry shown. But the analogy is not as far reaching as with the sphere, as it is not possible to embed a full 3D surface of constant negative curvature in a 4D space so what is shown is a local analogy.

In this universe a triangle would be less than 180 degrees.