Radio Science, Volume 15, Number 6, pages 1163–1170, November–December 1980

Minimum redundancy linear arrays for a large number of antennas

Masato Ishiguro[1]

The Research Institute of Atmospherics, Nagoya University, Toyokawa 442, Japan

(Received January 28, 1980; revised June 10, 1980; accepted June 11, 1980.)

A linear array that achieves maximum resolution for a given number of antennas is advantageous in earth-rotation aperture synthesis. This type of array is called a minimum redundancy linear array; it is obtained by reducing redundant spacings present in the array. Various methods are examined to find optimum arrays for a large number of antennas. From the systematic analysis of regular patterns in these arrays, a possibility of generalizing optimum arrays is suggested. Optimum configurations could also be used for distributing stations in an aperture synthesis array.

1. INTRODUCTION

The aperture synthesis technique has been widely used in radio astronomy to obtain high-resolution maps of radio sources [*Ryle,* 1975]. In this technique, it is required to sample as many independent spatial frequencies (or Fourier components) of the brightness distribution as possible. A linear array that achieves maximum resolution for a given number of antennas is advantageous in earth-rotation aperture synthesis by a single linear array. This type of array is called a minimum redundancy linear array (hereinafter called MRLA), which is obtained by reducing the number of redundant spacings present in the array.

When the number N of antennas is less than five, zero-redundancy linear arrays exist, which sample each spatial frequency only once at uniform intervals of unit spacing up to the maximum spacing. Antennas of the four-element zero-redundancy array [*Arsac,* 1955] are arranged as {1100101} (ones, occupied sites; zeros, unoccupied sites), and it provides six nonredundant spacings.

It is not easy to search out optimum MRLA configurations for a large number of antennas. *Bracewell* [1966] proposed a systematic arrangement of antennas. It is summarized as follows:

1. For an odd number of antennas ($N = 2m + 1$),

$$\underbrace{.1.\ \cdots\ .1.}_{m+1\ \text{antennas}}\ (m+2)\ \underbrace{.(m+1).\ \cdots\ .(m+1).}_{m\ \text{antennas}}$$

where points and numbers represent the positions of antennas and the spacings between them, respectively. The maximum spacing L of this array is

$$L = (m+1)^2 = (N+1)^2/4 \tag{1}$$

The redundancy R is defined as the ratio of the number of possible pairs of antennas to L:

$$R = {}_N C_2 / L = 2N(N-1)/(N+1)^2 \tag{2}$$

2. For an even number of antennas ($N = 2m$),

$$\underbrace{.1.\ \cdots\ .1.}_{m\ \text{antennas}}\ (m+1)\ \underbrace{.m.\ \cdots\ .m.}_{m\ \text{antennas}}$$

$$L = m(m+1) = N(N+2)/4 \tag{3}$$

$$R = 2(N-1)/(N+2) \tag{4}$$

The values of R for (2) and (4) approach $R = 2$ for a large value of N. An array of five antennas arranged in this way has been actually constructed at Stanford University [*Bracewell et al.,* 1973].

In the theory of numbers, a set of integers $\{n_1, n_2, \cdots, n_k\}$ is called a difference basis with respect to L if every positive integer n such that $0 < n \leqq L$ can be represented in the form of $n = n_i - n_j$. If $\{n_1, n_2, \cdots, n_k\}$ are considered as the positions of antennas, the problem of finding out optimum MRLA's may be interpreted as a problem of making difference bases that minimize k or maximize L. *Leech* [1956] examined this identical problem and gave some optimum solutions for $N \leqq 11$. There may be more than one optimum solution for a given N and L. This problem was discussed by *Moffet* [1968] with special interest in the ap-

[1] Present address: Tokyo Astronomical Observatory, University of Tokyo, Mitaka, Tokyo 181, Japan.

Paper number 80S0778.
0048-6604/80/1112-0778$01.00

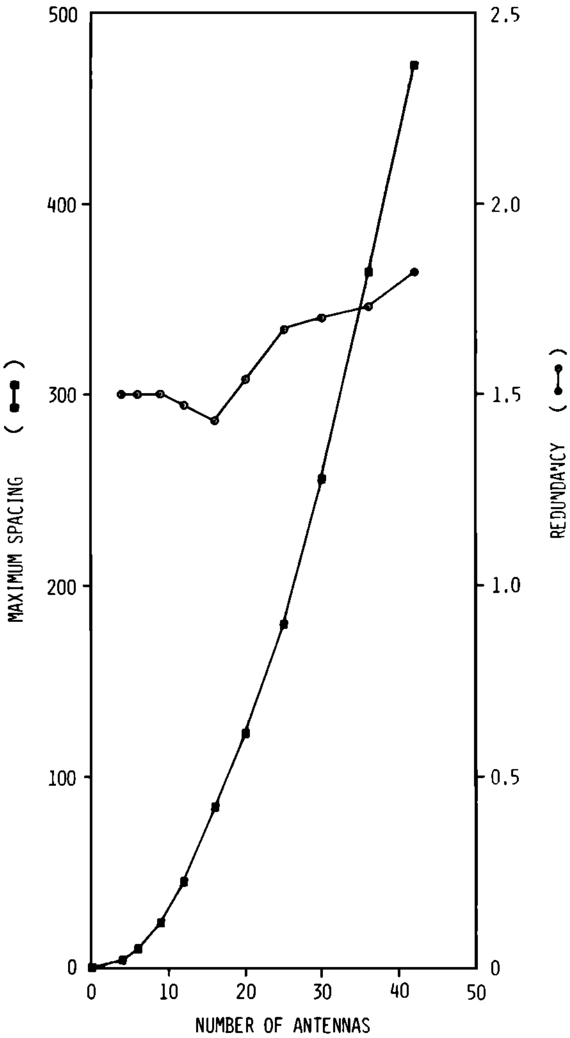

Fig. 1. MRLA's obtained by a combination of two MRLA's (see Table 1).

plication to aperture synthesis. The optimum solutions obtained by Leech have fairly small values of redundancy compared to those of Bracewell's arrangements. Therefore it can be said that optimum MRLA's for $N \leqq 11$ have already been worked out.

In this paper, we examine some methods to find out MRLA's for a large number of antennas and discuss advantages and disadvantages in these methods. In section 2, we describe a method to compose large MRLA's by a recursive use of optimum small MRLA's. In section 3, three methods are examined to find out MRLA's for an arbitrary number of antennas with the aid of a digital computer. In section 4, we analyze a regularity in array configurations obtained by these methods and propose a method of arranging antennas in a minimum-redundant manner by taking advantage of this regularity.

2. A METHOD TO COMPOSE LARGE MRLA'S BY A RECURSIVE USE OF OPTIMUM SMALL MRLA'S

It is possible to compose larger MRLA's by a minimum-redundant arrangement of small MRLA's by considering these small arrays as constituent elements for the new array. MRLA's for a large number of antennas are constructed by a recursive use of this process. This operation corresponds to suppressing the grating lobes of the array by narrowing the primary beam pattern and is explained by the principle of pattern multiplication in antenna theory. In this section, the properties of this method are described.

2.1 *A Combination of Two MRLA's*

Suppose that MRLA's of n antennas (MRLA 1) are arranged in the array configuration of an MRLA of m antennas (MRLA 2). In this case, the total number of antennas is

$$l = nm \qquad (5)$$

If the maximum spacings of MRLA 1 and MRLA 2 are N and M multiples of their respective unit spacings, the maximum spacing of the MRLA newly

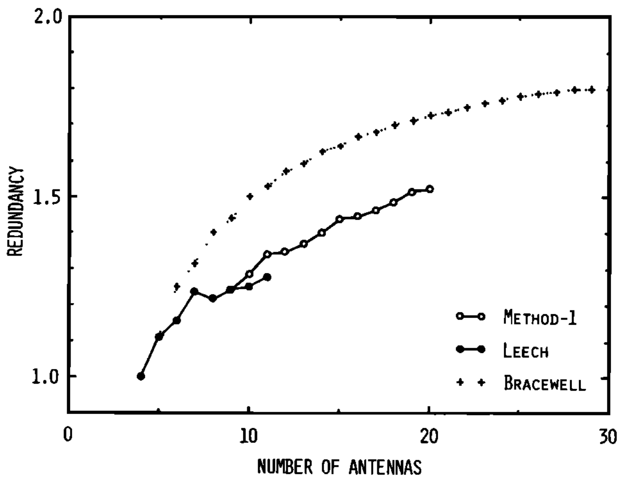

Fig. 2. Minimum redundancies of optimum MRLA's obtained by method 1 (see Table 4 and text).

synthesized is,

$$L = [2(N/2) + N + 1]\,M + 2(N/2) = 2NM + N + M \tag{6}$$

since the unit spacing of MRLA 2 is $2N + 1$ times that of MRLA 1. Equations (5) and (6) show that n and m (or N and M) are interchangeable in l (or L), so that the same result is obtained when MRLA 1 and MRLA 2 are interchanged.

The results for $L < 500$ are shown in Table 1 and Figure 1. A difference basis $\{n_1, n_2, \cdots, n_k\}$ such that $0 = n_1 < n_2 < \cdots < n_k = n$ is called a restricted difference basis. The MRLA configurations for up to seven antennas are quoted from the restricted difference bases in Leech's paper. The redundancy R in Table 1 is defined as

$$R = {}_lC_2/L = l(l-1)/2L \tag{7}$$

It is clear from Figure 1 that the combination of $n = 4$ and $m = 4$ gives the lowest redundancy ($= 1.43$) obtained with the method for $L < 500$.

2.2. *A Recursive Use of MRLA's*

If the array configuration of MRLA 2 is recursively used k times, the total number of antennas and the maximum spacing are, respectively,

$$l_k = m l_{k-1} \qquad (k \geqq 2) \tag{8}$$

$$L_k = 2L_{k-1}M + L_{k-1} + M$$

$$= (2M+1)L_{k-1} + M \qquad (k \geqq 2) \tag{9}$$

where $l_1 = n$ and $L_1 = N$. Equations (8) and (9) can be expressed in terms of l_1 and L_1, and the following expressions are obtained:

$$l_k = m^{k-1} l_1 = m^{k-1} n \qquad (k \geqq 2) \tag{10}$$

TABLE 1. A combination of two MRLA's.

n	N	m	M	l	L	R
2	1	2	1	4	4	1.50
2	1	3	3	6	10	1.50
3	3	3	3	9	24	1.50
3	3	4	6	12	45	1.47
4	6	4	6	16	84	1.43
4	6	5	9	20	123	1.54
5	9	5	9	25	180	1.67
5	9	6	13	30	256	1.70
6	13	6	13	36	364	1.73
6	13	7	17	42	472	1.82

TABLE 2. A recursive use of MRLA's for $n = m = 3$.

k	l_k	L_k	R_k
1	3	3	1.00
2	9	24	1.50
3	27	171	2.05

$$L_k = [(2M+1)^{k-1}(2L_1+1) - 1]/2$$

$$= [(2M+1)^{k-1}(2N+1) - 1]/2 \qquad (k \geqq 2) \tag{11}$$

When $n = m$ and $N = M$, (10) and (11) are simplified as

$$l_k = m^k \qquad (k \geqq 2) \tag{12}$$

$$L_k = [(2M+1)^k - 1]/2 \qquad (k \geqq 2) \tag{13}$$

The redundancy R_k for this case is from (7):

$$R_k = l_k(l_k - 1)/2L_k$$

$$= m^k(m^k - 1)/[(2M+1)^k - 1] \qquad (k \geqq 2) \tag{14}$$

For $k \gg 1$,

$$R_k \simeq [m^2/(2M+1)]^k \tag{15}$$

and the redundancy tends to increase monotonically with increasing k.

For simplicity, only the cases for $n = m$ are examined in the following discussion. As the zero-redundancy arrays are limited to the cases (1) $m = 3$, $M = 3$; and (2) $m = 4$, $M = 6$, with the exception of the trivial case of $m = 2$, $M = 1$, we examined these two cases:

Case 1: $m = 3$, in the limit of large k:

$$R_k \simeq (1.29)^k \tag{16}$$

while the results for $k = 1$, 2, and 3 are shown in Table 2.

Case 2: $m = 4$, in the limit of large k:

$$R_k \simeq (1.23)^k \tag{17}$$

while the results for $k = 1$, 2, and 3 are shown in Table 3.

The results for $k = 2$ in cases 1 and 2 correspond to the cases of a combination of two MRLA's. It is clear from the comparison of Tables 1 and 2 that case 2 is superior to case 1 in redundancy. In addition to cases 1 and 2, we examined two extra cases for $n = 4$, $m = 3$ and $n = 3$, $m = 4$, the results of which lie between those for cases 1 and 2. It is concluded that optimum solutions are obtained for case 2.

TABLE 3. A recursive use of MRLA's for $n = m = 4$.

k	l_k	L_k	R_k
1	4	6	1.00
2	16	84	1.43
3	64	1098	1.84

3. METHODS TO FIND MRLA'S FOR AN ARBITRARY NUMBER OF ANTENNAS WITH THE AID OF A DIGITAL COMPUTER

The method described in section 2 is applicable only to specific number of antennas. In this section, we examine three methods to find MRLA's for an arbitrary number of antennas with the aid of a digital computer. These three methods have the following process in common: For a given maximum spacing, these methods try to find arrays that can provide all the spacings from unit spacing up to the maximum. Those arrays that have minimum numbers of antennas are registered as candidates for MRLA's. This process is repeated with increasing maximum spacing stepwise in unit spacing. Thus optimum MRLA's that give maximum spacing for a given number of antennas are selected from these candidates.

3.1. *A Search Method by Use of Random Numbers*

A large number of random array configurations are generated in a digital computer and are examined to see whether they are to be rejected or not as favourable candidates for MRLA's. We used two methods of generating random array configurations. In one method, one starts with a configuration with all sites unoccupied and then places antennas at random in the unoccupied sites until the condition of full spacing is obtained. In the other method, one starts with a configuration with all sites occupied and then removes antennas at random from the occupied sites until the condition of full spacing is broken.

In our experience, the latter gives smaller values of redundancy than the former. Random numbers used in this approach are sampled from the uniform probability distribution over the whole length of the array and from a modified Gaussian distribution that shows higher probability at the both ends of the array than at the center. The algorithm is extremely simple, but it consumes computing time to little purpose. The redundancy of MRLA's thus obtained is not so small as expected.

3.2. *A Method to Search Out Some Restricted Possibilities (Method 1)*

As the number of ways of placing antennas, say, 20, in all the possible positions in the array of redundancy of 1.5 will amount to 10^{20}, it is prohibitive to search out all the possibilities. However, if the combinations are restricted by introducing some definite principles in placing antennas, it is not unrealistic to search out all the possibilities involved in them. Method 1 is an example of such an approach. The restriction introduced in method 1 is as follows.

If the maximum spacing is L, the number K of possible sites is $L + 1$. However, the first three antennas are automatically arranged so that the spacings L and $L - 1$ exist. In other words, one can start with the configuration of $\{.1.(L - 1).\}$. If the total number of antennas is N, $K - 3$ unoccupied sites are prepared for $N - 3$ antennas.

TABLE 4. Optimum MRLA's obtained by method 1.

N	L	R	Array Configuration
4	6	1.000	.1.3.2.
5	9	1.111	.1.3.3.2.
			.1.1.4.3.
6	13	1.154	.1.5.3.2.2.
			.1.3.1.6.2.
			.1.1.4.4.3.
7	17	1.235	.1.7.3.2.2.2.
			.1.3.6.2.3.2.
			.1.1.4.4.4.3.
			.1.1.1.5.5.4.
8	23	1.217	.1.3.6.6.2.3.2.
9	29	1.241	.1.3.6.6.6.2.3.2.
10	35	1.286	.1.3.6.6.6.6.2.3.2.
			.1.3.1.11.2.7.2.6.2.
11	41	1.342	.1.5.8.8.8.2.2.3.2.2.
			.1.3.6.6.6.6.6.2.3.2.
12	49	1.347	.1.5.8.8.8.8.2.2.3.2.2.
13	57	1.368	.1.5.8.8.8.8.8.2.2.3.2.2.
14	65	1.400	.1.5.8.8.8.8.8.8.2.2.3.2.2.
15	73	1.438	.1.5.8.8.8.8.8.8.8.2.2.3.2.2.
			.1.7.10.10.10.10.10.2.2.2.3.2.2.2.
16	83	1.446	.1.7.10.10.10.10.10.10.2.2.2.3.2.2.2.
17	93	1.462	.1.7.10.10.10.10.10.10.10.2.2.2.3.2.2.2.
18	103	1.485	.1.7.10.10.10.10.10.10.10.10.2.2.2.3.2.2.2.
19	113	1.513	.1.7.10.10.10.10.10.10.10.10.10.2.2.2.3.2.2.2.
			.1.9.12.12.12.12.12.12.12.2.2.2.2.3.2.2.2.2.
20	125	1.520	.1.9.12.12.12.12.12.12.12.12.2.2.2.2.3.2.2.2.2.

The total number n_T of these possibilities is

$$n_T = (K-3)!/\{[(K-3)-(N-3)]!(N-3)!\}$$
$$= (L-2)!/[(L-N+1)!(N-3)!] \quad (18)$$

For $N = 20$ and $L = 128$, $n_T \simeq 4.6 \times 10^{20}$.

As, in general, a larger spacing may be obtained in fewer ways than a smaller spacing, it will be advantageous to examine larger spacings first. In method 1, an antenna is placed so that the largest missing spacing may be obtained by a combination of this antenna with an antenna at one end (1 or K). Thus there are alternative choices in realizing a missing spacing from the left or right ends. The total number of possibilities in this methods is then

$$n_C = 2^{N-3} \quad (19)$$

For $N = 20$, $n_C \simeq 1.3 \times 10^5$. It is clear that the number of tries is drastically reduced in method 1. It is expected from (19) that the computing time will be doubled with an increase of 1 in N.

The results for $N \leqq 20$ are shown in Table 4 and Figure 2, together with those obtained by Leech and by Bracewell. For $N \leqq 9$, method 1 gives identical values of minimum redundancy with those obtained by Leech, and for $N = 10, 11$, it gives slightly greater values. It is an outstanding characteristic that optimum solutions are obtained when L is odd except for $L = 6$.

When unit spacing is close to the diameter of antennas and the angle of elevation is small in the direction of array baseline, there arises a practical problem of one antenna being shadowed by its neighbor. A modified version of method 1 is used to search out optimum MRLA's that have no unit spacings. One starts with the configuration of {.2.(L − 2).}. consequently, spacing $L-1$ is not allowed to exist in the array. In addition, sites 1, 3, and K are occupied, and sites 2, 4, $K-1$ are forced to be unoccupied so that $K-6$ unoccupied sites are prepared for $N-3$ antennas. The total number m_T of these possibilities is

$$m_T = (K-6)!/\{[(K-6)-(N-3)]!(N-3)!\}$$
$$= (L-5)!/[(L-N-2)!(N-3)!] \quad (20)$$

The ratio of m_T to n_T is

$$m_T/n_T = [(L-5)!(L-N+1)!]$$
$$\div [(L-2)!(L-N-2)!] = [(L-N+1)(L-N)$$
$$\cdot (L-N-1)]/[(L-2)(L-3)(L-4)] \quad (21)$$

The ratio m_T/n_T is always less than unity, and $m_T/n_T \simeq 0.64$ for $N = 20$, $L = 128$. This means the reduction in computing time is expected for the modified version of method 1.

The results are shown in Table 5 and Figure 3. In this case, since it is impossible to satisfy the condition of full spacing, the redundancy R is redefined as follows:

$$R = {}_NC_2/\text{number of independent spacings}$$
$$= N(N-1/[2(L-2)] \quad (22)$$

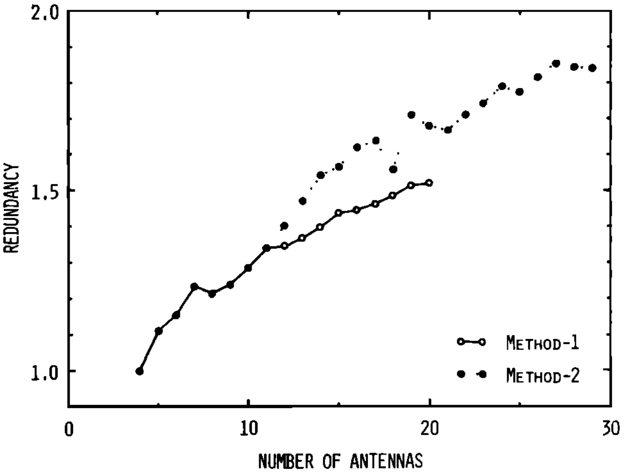

Fig. 4. Minimum redundancies of optimum MRLA's obtained by method 2.

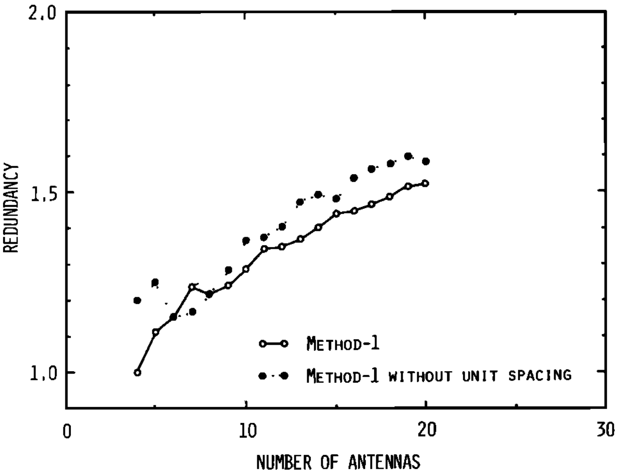

Fig. 3. Minimum redundancies of optimum MRLA's without unit spacing obtained by the modified version of method 1 (see Table 5).

TABLE 5. Optimum MRLA's without unit spacing obtained by the modified version of method 1.

N	L	R	Array Configuration
4	7	1.200	.2.2.3.
5	10	1.250	.2.2.3.3.
6	15	1.154	.2.2.5.3.3.
7	20	1.167	.2.2.5.5.3.3.
8	25	1.217	.2.2.5.5.5.3.3.
9	30	1.286	.2.2.5.5.5.5.3.3.
10	35	1.364	.2.2.5.5.5.5.5.3.3. .2.2.2.2.5.7.5.7.3.
11	42	1.375	.2.2.2.2.5.9.5.5.7.3.
12	49	1.404	.2.2.8.5.2.5.11.5.3.3.3.
13	55	1.472	.2.2.11.5.2.8.8.5.3.3.3.3. .2.2.11.2.5.3.11.7.3.3.3.3. .2.2.2.10.2.2.2.5.8.11.3.3.3. .2.2.2.2.2.7.7.5.11.3.9.3. .2.2.2.2.2.5.9.5.9.5.9.3.
14	63	1.492	.2.2.11.5.2.8.8.8.5.3.3.3.3. .2.2.5.2.2.5.5.12.3.5.14.3.3. .2.2.2.2.2.2.7.9.5.13.3.11.3. .2.2.2.2.2.2.5.11.5.11.5.11.3.
15	73	1.479	.2.2.2.2.12.5.11.5.5.5.9.3.7.3.
16	80	1.538	.2.2.2.2.2.14.5.8.5.7.5.11.3.9.3.
17	89	1.563	.2.2.2.2.2.2.2.7.11.5.13.5.15.3.13.3. .2.2.2.2.2.2.2.5.13.7.13.3.13.5.13.3. .2.2.2.2.2.2.2.5.13.5.13.5.13.5.13.3.
18	99	1.577	.2.2.2.10.5.2.2.26.3.7.3.5.12.7.3.5. .2.2.2.2.2.2.2.2.7.13.5.15.5.17.3.15.3. .2.2.2.2.2.2.2.2.5.15.7.15.3.15.5.15.3. .2.2.2.2.2.2.2.2.5.15.5.15.5.15.5.15.3.
19	109	1.598	.2.2.20.5.2.20.5.5.14.8.5.3.3.3.3.3.3.3. .2.2.2.2.12.2.2.2.5.10.15.10.5.10.15.3.7.3. .2.2.2.2.2.2.2.18.5.17.5.9.2.5.15.3.13.3. .2.2.2.2.2.2.2.2.2.7.15.5.17.5.19.3.17.3. .2.2.2.2.2.2.2.2.2.5.17.7.17.3.17.5.17.3. .2.2.2.2.2.2.2.2.2.5.17.5.17.5.17.5.17.3.
20	122	1.583	.2.2.2.2.2.2.16.5.15.7.7.11.5.9.18.3.11.3.

It is clear from Figure 3 that MRLA's without unit spacing have larger values of redundancy than usual MRLA's, except for the case $N = 7$. Although for $N = 6$ and $N = 8$, both types of arrays give the same minimum redundancy; MRLA's without unit spacing are superior in resolution.

3.3. *A Method to Search Out Some Restricted Possibilities (Method 2)*

In this method, the largest missing spacing is examined not only from both ends of the array but also from all the existing antennas previously placed. It is the same as in method 1: to start with the configuration of $\{.1.(L-1.\}$ and to examine larger spacings preferentially. A site is selected as optimum which, if occupied, gives as many missing spacings as possible. When more than one site is selected as optimum at some stage, they are registered without exception to examine all the combinations of tree structure derived from them. This process is repeated until the condition of full spacing is obtained. Those configurations that give minimum redundancy are registered as candidates for optimum MRLA's.

The values of minimum redundancy obtained for $N \leqq 29$ by method 2 are shown in Figure 4. For $N \leqq 11$, both methods give the same results, and for $N \geqq 12$, method 2 is inferior in redundancy to method 1. However, it takes 10^4 times less computing time in method 2 than in method 1. Figure 5 shows the results when optimum MRLA's without unit spacing are searched by a modified version of method 2.

4. A REGULARITY IN OPTIMUM MRLA'S

There are apparent regular patterns in the configurations of optimum MRLA's for a large value of N. It is clear that the largest spacing between succesive pairs of antennas repeats many times at the central part of the array.

For example, optimum MRLA's obtained by method 1 for large N could be generalized as follows:

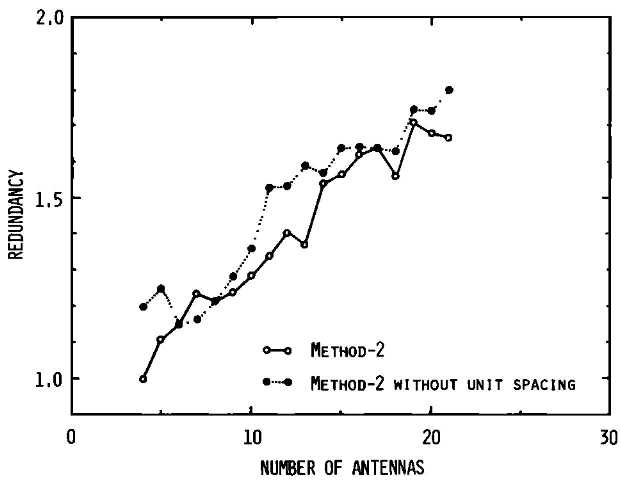

Fig. 5. Minimum redundancies of optimum MRLA's without unit spacing obtained by the modified version of method 2.

TABLE 6. Optimum MRLA's with a regularity.

N	L	Array Configuration	Reference*
$n + 3$	$3n + 3$	.1.3. ···· .3.2.	A
$n + 4$	$4n + 5$	.1.1.4. ···· .4.3. .1.(2n + 1).3.2. ···· .2.	B
$n + 5$	$5n + 7$	.1.1.1.5. ···· .5.4.	C
$n + 6$	$6n + 11$	.1.3.6. ···· .6.2.3.2.	D
$n + 7$	$7n + 15$	.1.2.3.7. ···· .7.4.4.1.	E
$n + 8$	$8n + 17$	.1.5.8. ···· .8.2.2.3.2.2.	F
$n + 9$	$9n + 20$	.1.2.5.9. ···· .9.4.2.4.1.1. .1.1.2.4.9. ···· .9.5.5.1.1.	G
$n + 10$	$10n + 23$	.1.7.10. ···· .10.2.2.2.3.2.2.2.	H
$n + 11$	$11n + 28$	.1.3.2.1.3.11. ···· .11.8.4.1.3.2.	I
$n + 12$	$12n + 29$	.1.9.12. ···· .12.2.2.2.2.3.2.2.2.2.	J
$n + 14$	$14n + 35$	.1.11.14. ···· .14.2.2.2.2.2.3.2.2.2.2.2.	K
$n + 16$	$16n + 41$	.1.13.16. ···· .16.2.2.2.2.2.2.3.2.2.2.2.2.2.	L

*See Figure 6.

$$\underbrace{.1.p.}_{3\text{ antennas}}\ \underbrace{(p+3)\ .\cdots.\ (p+3)}_{n-1\text{ antennas}}\underbrace{.2.\cdots.2.3.2.\cdots.2.}_{p+1\text{ antennas}}$$

$$(n = 1, 2, \cdots)$$

where p is an odd number. The necessary number of antennas, the maximum spacing, and the redundancy are, respectively,

$$N = 3 + (n - 1) + (p + 1) = n + p + 3 \quad (23)$$

$$L = (p + 3)n + 3p + 2 \quad (24)$$

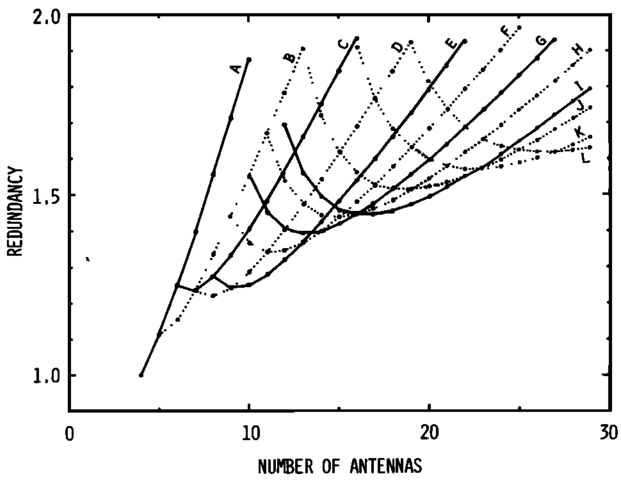

Fig. 6. Minimum redundancies of optimum MRLA's with regularity.

$$R = N(N - 1)/\{2[(p + 3)N - (p^2 + 3p + 7)]\} \quad (25)$$

Array configurations for $p = 1, 3, \cdots, 13$ are shown in Table 6 as a class of arrays for $N = n +$ (even number). The corresponding values of redundancy are illustrated in Figure 6 by dotted lines. It is clear from Figure 6 that the values of minimum redundancy of optimum solutions in method 1 lie just on the lower envelope of those given by (25).

MRLA's with regularity exist also for $N = n +$ (odd number), the configurations of which are shown in Table 6 and Figure 6. Arrays for $N = n + 9$, $n + 11$ are obtained by a heuristic method with trial and error. It is easily proved by mathematical induction that these arrays have all spacings from unit spacing up to the maximum. As is shown in Figure 6, arrays for $N = n +$ (odd number) are superior in redundancy to those for $N = n +$ (even number). Probably such arrays will exist for $N \geqq n + 13$, though we have never tried to find them.

5. SUMMARY AND CONCLUSION

Various methods were examined to find optimum MRLA's for a large number of antennas. In the first method, larger MRLA's are obtained by using small MRLA's recursively as constituent elements for the new array. If we restrict ourselves to use only zero-redundancy arrays as small MRLA's, the four-antenna, zero-redundancy array is advantageous. As a result, when $l = 16$ and $L = 84$ ($n = m = 4$ and $N = M = 6$), a minimum redundancy $R = 1.43$ was obtained, which is also the minimum value found by any of the methods described.

Secondly, three methods were developed to find MRLA's for an arbitrary number of antennas with the aid of a digital computer. These are heuristic methods by use of random numbers, a method in which the largest missing spacing is examined from the ends of array (method 1) and a method in which the largest missing spacing is examined not only from the ends but also from all the existing antennas (method 2). The method using random numbers is simple but not so efficient. Method 1 gave solutions very close to the results obtained by Leech. Except for the computing time, method 2 was inferior to method 1 in spite of its complicated algorithm. With modified versions of these methods, optimum MRLA's without unit spacing were searched out in order to avoid the practical difficulty of one antenna being shadowed by its neighbor.

Finally, from the systematic analysis of regular patterns in optimum MRLA's, general forms of these MRLA's were summarized (Table 6). It is clear that the results obtained by method 1 correspond to the cases of the lowest possible redundancy for $N = n +$ (even number) and that arrays for $N = n +$ (odd number) are superior in redundancy to those for $N = n +$ (even number).

Leech gives bounds for the redundancy of restricted difference bases in the limit of large N of $1.217 \lesssim R \lesssim 1.674$. As the optimum MRLA's described in this paper have redundancies well above Leech's lower bound, while they are less than 1.674, it may still be possible to find optimum MRLA's of substantially lower redundancy.

Acknowledgment. The author wishes to express his sincere thanks to the reviewers for a critical reading of the manuscript and valuable suggestions.

REFERENCES

Arsac, J. (1955), Nouveau réseau pour l'observation radio astronomique de la brilliance sur le soleil à 9359 MC, *Compte. Rend. Acad. Sci., 240,* 942–945.

Bracewell, R. N. (1966), Optimum spacings for radio telescopes with unfilled apertures, *Nat. Acad. Sci. Nat. Res. Counc., Publ. 1408,* 243–244.

Bracewell, R. N., R. S. Colvin, L. R. D'Addario, C. J. Geben Kemper, K. M. Price, and A. R. Thompson (1973), The Stanford five-element radio telescope, *Proc. IEEE, 61,* 1249–1257.

Leech, J. (1956), On the representation of 1, 2, $\cdots$, n by differences, *J. London Math. Soc., 31,* 160–169.

Moffet, A. T. (1968), Minimum-redundancy linear arrays, *IEEE Trans., AP-16,* 172–175.

Ryle, M. (1975), Radio telescope of large resolving power, *Science, 188,* 1071–1083.