NONLINEAR SIGNAL PROCESSING AND NONUNIFORM LINEAR ANTENNA ARRAY DESIGN FOR DOA ESTIMATION OF COHERENT SOURCES

Nicholas K. Spencer

Cooperative Research Centre for Sensor Signal and Information Processing (CSSIP),
SPRI Building, Technology Park, Mawson Lakes, South Australia, 5095, Australia
nspencer@cssip.edu.au

ABSTRACT

This paper discusses one solution technique for an optimisation problem involving nonuniform linear array (NLA) geometry. This NLA design problem seeks an optimal geometry from the class of fully-augmentable arrays (*ie*. those with a complete set of intersensor differences) created to support the generalised spatial smoothing (GSS) algorithm, whose aim is to resolve the directions-of-arrival (DOA's) of some number of fully correlated signals. This problem is identified as an integer multicriteria optimisation (MO) problem. Presented results indicate that significant improvements in DOA estimation accuracy may be achieved by such optimised geometries compared with uniform or minimum-redundancy arrays.

1. BACKGROUND

Modern array signal processing applications place great demands on the underlying antenna array geometry to deliver high performance. The history of major developments in array geometry design mirrors the history of achievements in numerical optimisation. As early as the 1940's, the pioneering contributions of Dolf [8], and Woodward and Lawson [21] in array design were formulated as the solution to optimisation problems, and each successive breakthrough in mathematical programming has been closely followed by its associated application to array design.

During these early years, only conventional single antennas ("dishes") were implemented in practice, and hence optimisation efforts at first concentrated on the current distribution function. Later, when the first antenna arrays were assembled, it was immediately realised that the geometric distribution of individual sensors along a line (linear arrays) or across some area (planar arrays) was crucial to the optimisation process as well.

The first attempts to optimise array geometry considered conventional beamforming. While these problems were formulated in terms of integer programming, later developments in dynamic programming were used to find solutions, such as in the work of Skolnik, Nemhauser and Sherman [18]. For those scientists mainly involved with radar and communications applications, minimum beam-pattern sidelobe level was a major goal in these optimisation studies.

After the important conjecture made by Arsac [6], who may have been the first engineer to discuss the unique optimal 4-element geometry $\boldsymbol{d} = [0,\, 1,\, 4,\, 6]$, Moffet [14] and others elaborated on Arsac's work, incorporating certain results from number theory (*eg*. Leech [13]). These papers properly formulated the integer programming problem for nonredundant and minimum-redundancy antenna array design, though strictly optimal solutions were obtained only by exhaustive search, despite numerous studies.

After the famous publications of Burg [7] and Pisarenko [16], it still took some time for the research community to recognise that the minimum-redundancy criterion is appropriate for NLA geometry optimisation when nonlinear super-resolution techniques are involved (*eg*. [12]). Since then, studies into the area of joint array geometry and signal processing optimisation have continued (*eg*. [11]), heavily influencing the ideology of passive direction-finding (DF) systems.

Today, the theory of optimal geometry and processing for DF applications is far from complete. A recently tackled problem concerns joint optimal geometry and processing for DF involving fully correlated sources. Standard techniques involve various modifications of the spatial smoothing idea, which originally applied only to a uniform linear array (ULA) [17]. The area of joint array geometry and signal processing design optimisation for coherent sources was sketched in [4]. We now more fully discuss the optimisation of NLA geometry with respect to embedded "partial arrays", in order to facilitate improved DOA estimation accuracy for a given maximum number of coherent signals.

2. PARTIAL ARRAYS

Suppose that we have M antenna sensors and wish to construct the "optimal" NLA. Further suppose that the sensor positions $\boldsymbol{d} \equiv [0,\, d_2,\, d_3,\, \ldots,\, d_M]$ are restricted to integer values (usually measured in half-wavelength units). Here "optimal" means allowing the most accurate DOA estimate possible. In principle, we could use the Cramér–Rao bound to determine the best NLA for any given (fixed) source configuration [10]; however the number of signals and their (probably changing) angular separations are *a priori* unknown. We need to define the optimisation problem and criteria from a different perspective.

We consider the DOA estimation problem for some small pre-specified maximum number of coherent signals m (of arbitrary configuration) using the special class of *partial-array* NLA geometries and the corresponding *generalised spatial smoothing* (GSS) algorithm. While full details of the GSS method can be found in [3, 4], the main relevant point is that GSS delivers greater DOA estimation accuracy for NLA's which are rich in a variety of embedded partial arrays. We also need to define a mathematical cost function which can be used as the basis of optimisation. Since the signal configuration is arbitrary, we will invent a cost function which attempts to encapsulate the overall "goodness" of any particular NLA for all possible signal scenarios, based on the properties of the embedded partial arrays.

Let the *co-sequence* of an array $\boldsymbol{d}$ be its set of $M-1$ consecutive intersensor separations (*ie*. differences), while its *co-array* is the sorted set of $M(M-1)/2$ differences. We define a *partial array* to be a group of nonuniform linear noncontiguous sub-arrays of identical co-sequence structure [4]. Associated with each partial array are its *multiplicity* κ (number of occurrences or instances), *order* ℓ (number of co-sequence elements involved), and aperture a. A given NLA will have n embedded partial arrays, with a total of N instances. The GSS technique may be applied to a NLA providing it yields at least one partial array of multiplicity $\kappa \geq m$ and order $\ell \geq m$, where m is the number of fully correlated signals.

The following simple example of a partial array will illustrate these properties. The NLA

$$\boldsymbol{d}_{eg} = [0,\, 1,\, 5,\, 6,\, 9,\, 11,\, 12] \tag{1}$$

has the co-sequence

$$\boldsymbol{c}_{eg} = [1,\, 4,\, 1,\, 3,\, 2,\, 1] \tag{2}$$

in which is embedded the partial array with co-sequence structure

$$\boldsymbol{c}_1 = [1,\, 5] \tag{3}$$

since this co-sequence repeatedly occurs as a fixed sub-array pattern of the original array elements as follows:

$$\begin{aligned} \boldsymbol{d}_{11} &= [0,\, 1,\, 6] \\ \boldsymbol{d}_{12} &= [5,\, 6,\, 11] \\ \boldsymbol{d}_{13} &= [0,\, 5,\, 6] \\ \boldsymbol{d}_{14} &= [6,\, 11,\, 12]. \end{aligned} \tag{4}$$

This partial array has a multiplicity of $\kappa_1 = 4$. Note that the instances $\boldsymbol{d}_{13}$ and $\boldsymbol{d}_{14}$ exist as mirror-images (the co-sequence order is reversed). This partial array has order $\ell_1 = 2$, and aperture $a_1 = \sum_{j=1}^{\ell_1} \boldsymbol{c}_{1j} = 6$. The exhaustive list of partial arrays for the $\boldsymbol{d}_{eg}$ geometry is

$$\begin{array}{llll} \boldsymbol{c}_1 = [1,\, 5] & \kappa_1 = 4 & \ell_1 = 2 & a_1 = 6 \\ \boldsymbol{c}_2 = [1,\, 4] & \kappa_2 = 2 & \ell_2 = 2 & a_2 = 5 \\ \boldsymbol{c}_3 = [1,\, 6] & \kappa_3 = 2 & \ell_3 = 2 & a_3 = 7 \\ \boldsymbol{c}_4 = [1,\, 10] & \kappa_4 = 2 & \ell_4 = 2 & a_4 = 11 \\ \boldsymbol{c}_5 = [1,\, 11] & \kappa_5 = 2 & \ell_5 = 2 & a_5 = 12 \\ \boldsymbol{c}_6 = [5,\, 6] & \kappa_6 = 3 & \ell_6 = 2 & a_6 = 11 \\ \boldsymbol{c}_7 = [1,\, 5,\, 5] & \kappa_7 = 2 & \ell_7 = 3 & a_7 = 11 \\ \boldsymbol{c}_8 = [1,\, 5,\, 6] & \kappa_8 = 2 & \ell_8 = 3 & a_8 = 12 \end{array} \tag{5}$$

so that the number of embedded partial arrays is $n = 8$, with a total of $N = \sum_{j=1}^{n} \kappa_j = 19$ instances.

At this stage, embedded partial arrays are found by exhaustive computer search. The set of useful partial arrays may be found by searching over the range of values $m \leq \ell \leq M-2$ and $1 \leq c \leq d_M - \ell + 1$, where c is the possible value of any partial co-sequence element. Thus the number of candidate partial arrays to be searched is $O(d_M^{M-2})$. Since this can be an extremely time-consuming process, our searches are often conducted over a smaller range of values.

The GSS algorithm introduced in [4] consists of an initialisation step followed by local ML refinement. The initialisation step is based on the *PA-MUSIC* approach involving all appropriate partial arrays.

Suppose that an NLA yields a total of N partial arrays, each of multiplicity κ_i, order ℓ_i and aperture a_i ($i = 1, \ldots, N$). Let $\boldsymbol{y}_{ij}$ be a $(\ell_i + 1)$-variate snapshot vector corresponding to the j^{th} instance ($j = 1, \ldots, \kappa_i$) of the i^{th} partial array. If any instance of a partial array occurs as a mirror-image (*ie*. in reverse order), then the corresponding snapshot vector is observed by reversing the order of antenna samples and taking the complex conjugate of the vector. Thus for each partial array we may define the $(\ell_i + 1) \times (\ell_i + 1)$ partial array covariance matrix by spatial smoothing to be

$$\hat{R}_i \;=\; \sum_{j=1}^{\kappa_i} \boldsymbol{y}_{ij}\, \boldsymbol{y}_{ij}^H. \tag{6}$$

Let $\hat{G}_i$ be the noise eigen-subspace of $\hat{R}_i$, then $\hat{G}_i$ consists of at least one eigenvector (since $m \ll M$). The PA-MUSIC technique is:

$$\text{find} \quad \max_{\theta} f_{PA}(\theta) \;:=\; \min_{\theta} \sum_{i=1}^{N} \boldsymbol{a}_i^H(\theta)\, \hat{G}_i\, \hat{G}_i^H\, \boldsymbol{a}_i(\theta) \tag{7}$$

where $\boldsymbol{a}_i(\theta)$ is the $(\ell_i + 1)$-variate manifold ("steering") vector which corresponds to the given partial array geometry. Evidently this approach eliminates non-coinciding ambiguities.

Thus the effectiveness of DOA estimation delivered by GSS is directly related to the number, variety and $\kappa\ell a$–properties of the available partial arrays in the following ways. Firstly, we desire a NLA which contains as many partial arrays as possible for a given M, preferably each with large κ and ℓ. (Therefore neither nonredundant nor minimum-redundancy arrays are suitable; we need to identify a new class of NLA.) Secondly, since DOA estimation accuracy improves with increasing array aperture, the partial arrays should have apertures as long as possible. Note that if the NLA has a very large aperture, then its extreme sparsity means very few redundancies and hence few partial arrays, so that these two criteria are competitive. We constrain the solution to be a fully-augmentable array (having no missing numbers, or *gaps*, in the co-array), since such an array is inherently unambiguous [2].

The complete statement of the optimisation problem is that for a given number of sensors M and a given maximum number of fully correlated signals m whose directions are to be estimated, we are to distribute these M elements over the 1–D grid of integer values to create a fully-augmentable NLA, in the way which optimises DOA estimation performance, a quantity which cannot be calculated but which is approximated by trading-off the above two competitive criteria.

3. OPTIMISATION METHOD

We are dealing with a so-called constrained integer multicriteria (multiple-criteria or multi-objective) optimisation (MO) problem [9]. Since effective computational algorithms for finding the strictly optimal solution have not yet been developed, even for less sophisticated integer programming problems such as minimum-redundancy optimisation, and since an exhaustive search would be too time-consuming, we have developed the following optimisation problem cost function A and three-stage optimisation approach.

Consider the simple cost function

$$A(\boldsymbol{d}, \ell_{\min}, \ell_{\max}, c_{\min}, c_{\max}) = \sum_{j=1}^{n} a_j^2 \qquad (8)$$

which in some way incorporates both criteria, since it favours larger partial array apertures *and* a large number of different partial arrays. (Note that the above sum is only over all *suitable* partial arrays, *ie*. those with $\kappa \geq m$ and $\ell \geq m$.) By choosing such a cost function, we have collapsed the MO problem to a univariate optimisation problem.

In the first stage, we define the total antenna array aperture by selecting an initial nonredundant array geometry of M_1 elements, for example from the lists published in [19, 20]. A nonredundant array contains no partial arrays at all, but has maximum aperture for a given number of gaps. (Any nonredundant array with more than four elements must have gaps [14].) At this stage we have $(M{-}M_1)$ elements remaining at our disposal to distribute optimally amongst the vacant integer positions.

Secondly, we eliminate all gaps with the minimum possible number of elements M_2, using an exhaustive tree search technique. This results in an (often lengthy) list of gap-free (fully-augmentable) candidate arrays.

Thirdly, we utilise the remaining $M_3 = (M - M_1 - M_2)$ degrees of freedom by adding this number of elements to each candidate geometry in turn, searching for the array that maximises the selection criterion (cost function) $A(\boldsymbol{d})$. For each candidate, we conduct an integer programming search by adding elements to vacant positions one-by-one, at each stage simply selecting the maximum-A geometry for further investigation. This method will find the global optimum (of this third stage) provided that the problem is separable, *ie*. that each of the M_3 degrees of freedom is independent of the other; while this is not strictly true, the small level of interaction between successive introduced elements relegates this to a second-order effect.

By itself, this would be in principle a relatively straightforward optimisation problem, since for a given M, m, M_1, $\ell_{\min}$, $\ell_{\max}$, $c_{\min}$ and $c_{\max}$ we can compute the A-optimal NLA. However, the entire three-stage optimisation problem should now be wrapped inside an M_1-optimisation, since our initial choice of M_1 was somewhat arbitrary.

4. EXAMPLE RESULTS

The following examples illustrate array geometry optimisation results for an $M = 16$ element array. The initial choice $M_1 = 10$ gives us the starting-point 10-element nonredundant Sverdlik array [19]

$$\boldsymbol{d}_{55}^{(0)} = [0, 1, 6, 10, 23, 26, 34, 41, 53, 55]\,. \qquad (9)$$

The exhaustive tree search of stage two yields 37 candidate gap-free geometries, each with 14 elements and 36 redundancies. The integer programming maximisation of stage three finds that of these candidates,

$$\boldsymbol{d}_{55}^{(1)} = [0, 1, \underline{5}, 6, 10, 23, 26, 34, \underline{37}, 41, \underline{44}, \underline{52}, 53, 55] \qquad (10)$$

is best (in the search range $\ell = 3$ and $c \in [1, 18]$), since with the addition of two sensors it yields the 16-element NLA

$$\boldsymbol{d}_{55} = [0, 1, \underline{5}, 6, \underline{8}, 10, \underline{\underline{19}}, 23, 26, 34, \underline{37}, 41, \underline{44}, \underline{52}, 53, 55] \qquad (11)$$

having the maximal cost function $A = 38467$ (and 65 redundancies). Thus we have partitioned our $M = 16$ elements in this example by $\{M_1 = 10,\, M_2 = 4,\, M_3 = 2\}$. Note that this three-stage optimisation search took a few days computing time on a modern workstation, even with its rather modest search range. At this point, we have no alternative but to assume that any NLA rich in partial arrays for a restricted search set $\{\ell, c\}$ will be similarly superior for a more expansive set.

Indeed, Table 1(a) shows the $\kappa\ell$-distribution and Fig. 1(a) illustrates the a-distribution of partial arrays for $\boldsymbol{d}_{55}$ for the expanded search range $\ell \in [3, 5]$ and $c \in [1, 30]$, whence we find $A = 99441$. This array performs better than the 16-element ULA because of the large numbers of embedded partial arrays, each of significant aperture. The minimum-redundancy array of comparable total aperture ($M_\alpha = 58$) has 13 elements [15], so we could consider $\boldsymbol{d}_{55}$ to be a type of "optimal" solution by the introduction of only three additional elements to the minimum-redundancy structure.

A 16-element NLA with different partial array characteristics is illustrated by the shorter array

$$\boldsymbol{d}_{34} = [0, 1, 4, 5, 8, 9, 10, 14, 15, 16, 18, 22, 23, 25, 32, 34]\,. \tag{12}$$

Table 1(b) and Fig. 1(b) show its partial array $\kappa\ell a$-properties. While the maximum partial array aperture is here less than that of $\boldsymbol{d}_{55}$, the number of partial arrays is much greater. In any case, we would expect $\boldsymbol{d}_{34}$ to perform better than the 16-element ULA $\boldsymbol{d}_{15}$. Note that the 10-element optimal minimum-redundancy array (containing no suitable partial arrays) has a similar aperture ($M_\alpha = 36$) [15], so all 480 partial array instances are created by adding six sensors.

These examples illustrate NLA geometry design complementing a specific signal processing algorithm (GSS), where apertures are maximised by redundancy minimisation and partial array distributions are optimised by creating redundancies. The DOA estimation performance analysis conducted in [4] has shown that the final bearing estimation accuracy for GSS applied to the geometries optimised by this method is significantly superior to the conventional ULA geometry, using the standard spatial smoothing algorithm. In that paper, a comparison of error statistics is presented for DOA estimation simulations with three sources, using the geometries $\boldsymbol{d}_{55}$, $\boldsymbol{d}_{34}$ and the corresponding 16-element ULA. For this benchmark $\boldsymbol{d}_{15}$, a standard spatial smoothing technique was used with (in our nomenclature) the single partial array $\boldsymbol{c} = [1, \ldots, 1]$ of order $\ell = 13$, which has multiplicity $\kappa = 6$ (three of them mirror-images) and aperture $a = 13$.

5. EXTENSIONS OF THE METHOD

It should be noted that the above *ad hoc* procedure to find a close-to-optimal solution may be applied to other problems of a similar nature. Minimum-redundancy arrays, with their contiguous co-arrays, perfectly suit interferometric or covariance moment-based spectral (DOA) estimation methods. Some applications demand more than this single criterion; *eg*. suppose we also wish to minimise the NLA beam pattern sidelobe level within some angular range of the main lobe. We may do this by starting with minimum-redundancy arrays, introducing additional redundancies in order to decrease the array pattern sidelobes. Again, we have two competing criteria — maximal array aperture versus minimal maximum sidelobe levels within some range.

In fact, we may apply the same three-stage optimisation approach, simply replacing the cost function $A(\boldsymbol{d})$ by the appropriate calculation of the maximum sidelobe level within some pre-specified range. In this case, the third stage of our approach is essentially the same as the dynamic programming scheme of [18].

6. SUMMARY

We have considered a problem involving nonuniformly spaced linear array geometry optimisation, in the context of enhancing the performance of modern super-resolution techniques in spatial spectrum (DOA) estimation. This rigorous optimisation problem has been formulated in terms of a multicriteria integer programming problem, where efficient general computational schemes to find the globally optimum solution are unknown. The optimisation problem has been reduced to a simplified form where effective techniques can be applied, based on dynamic programming principles.

Optimisation efficiency in terms of spectral (DOA) estimation accuracy has been analysed elsewhere [1, 5, 4], and in most cases is found to be very high and significantly superior to the conventional ULA geometry coupled with standard MUSIC-type routines.

7. REFERENCES

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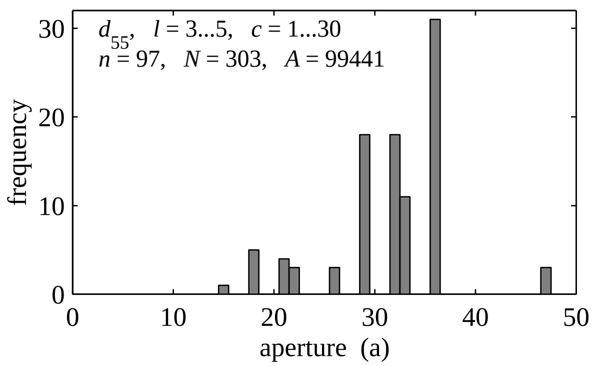
ℓ \ κ	3	4	5	6	7	8	9
3	40	9	0	0	0	0	0
4	33	3	0	0	0	0	0
5	12	0	0	0	0	0	0

ℓ \ κ	3	4	5	6	7	8	9
3	76	26	2	0	0	0	0
4	43	0	0	0	0	0	0
5	3	0	0	0	0	0	0

Table 1: Partial array distribution by multiplicity (κ) and order (ℓ) for the NLA geometry d_{55} (left) and d_{34} (right) for $m = 3$ and the search range $\ell \in [3, 5]$ and $c \in [1, 30]$.



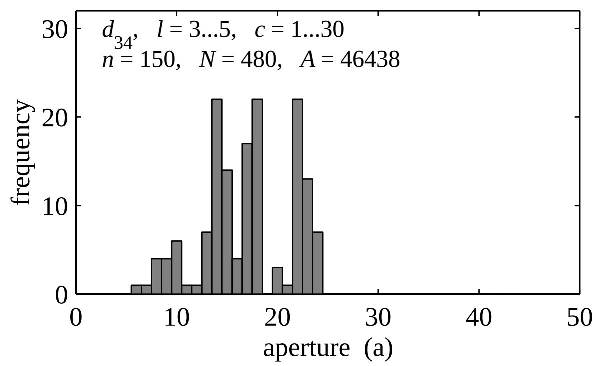


Figure 1: Aperture histogram of partial arrays embedded in d_{55} (left) and d_{34} (right).

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