PHY3145 Topics in Theoretical Physics

Astrophysical Radiation Processes

4: Relativistic effects II

Dr. J. Hatchell, Physics 407, J.Hatchell@exeter.ac.uk

Course structure

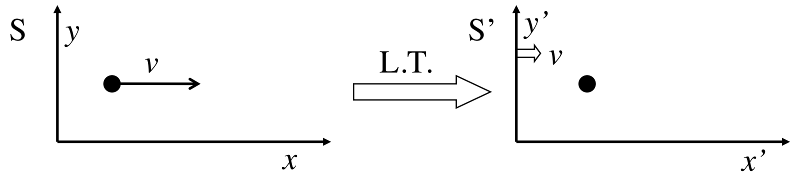
1. **Radiation basics**. Radiative transfer.
2. **Accelerated charges produce radiation**. Larmor formula. Acceleration in electric and magnetic fields – non-relativistic bremsstrahlung and gyrotron radiation.
3. **Relativistic modifications I**. Doppler shift and photon momentum. Thomson, Compton and inverse Compton scattering.
4. **Relativisitic modifications II**. Emission and arrival times. Superluminal motion and relativistic beaming. Gyrotron, cyclotron and synchrotron beaming. Acceleration in particle rest frame.
5. **Bremsstrahlung and synchrotron spectra.**

3145 Topics in Theoretical Physics - Radiation processes - Dr J Hatchell

Motion in the particle's rest frame

Aim: Calculate motion in particle's rest frame (usually so that we can apply Larmor formula)

Method: Lorentz Transform position, fields etc. from observer's frame S to comoving frame S' (standard relativity convention)



Observer's frame — Particle's rest frame

Examples

- Acceleration in E-field: Bremsstrahlung (relativistic) - HOMEWORK
- Acceleration in B-field: Cyclotron/Synchrotron

Example: Synchrotron particle motion and acceleration in rest frame

3145 Topics in Theoretical Physics - radiation processes - Dr J Hatchell

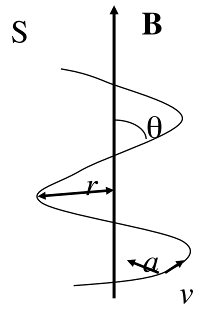
Aims:

i. Relativistic motion of charged particle in magnetic field;
ii. Acceleration in particle rest frame (needed for Larmor formula)
iii. Total power

Method:

i. Lorentz force in lab. frame
ii. Lorentz transform B into instantaneous rest frame. Use Lorentz force to calculate acceleration.
iii. Larmor's formula in particle rest frame

Synchrotron motion (continued…):



θ is the pitch angle (angle between B and v)

i. Relativistic motion of charged particle in magnetic field

Use Lorentz force in lab. frame

$$\frac{d}{dt}(\gamma_v m\mathbf{v}) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\gamma_v ma = qvB\sin\theta$$

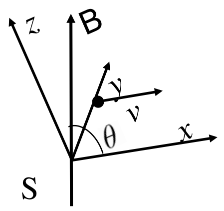
Equate with centripetal acceleration

$$\frac{{v_\perp}^2}{r} = \frac{v^2\sin^2\theta}{r} = \frac{qvB\sin\theta}{\gamma_v m}$$

Angular velocity – relativistic gyrofrequency

$$\boxed{\omega_r = \frac{v_\perp}{r} = \frac{qB}{\gamma_v m} = \frac{\omega_g}{\gamma_v}}$$

Synchrotron acceleration (continued…):



Instantaneously take v along x and B in x-z plane

$$\mathbf{B} = (B_x, 0, B_z) = (B\cos\theta, 0, B\sin\theta)$$
$$\mathbf{E} = (0, 0, 0)$$

ii. Acceleration in charge rest frame. Lorentz transform E and B and apply Lorentz force remembering that in rest frame $v' = 0$

$$\begin{aligned} E'_x &= E_x &&= 0 & B'_x &= B_x &&= B\cos\theta \\ E'_y &= \gamma(E_y - vB_z) &&= -\gamma vB\sin\theta & B'_y &= \gamma(B_y + v/c^2 E_z) &&= 0 \\ E'_z &= \gamma(E_z + vB_y) &&= 0 & B'_z &= \gamma(B_z - v/c^2 E_y) &&= \gamma B\sin\theta \end{aligned}$$

$\gamma'_v m\mathbf{a}' = q(\mathbf{v}' \times \mathbf{B}' + \mathbf{E}')$ only has non-zero terms in $\mathbf{E}'$

$$\boxed{a' = \frac{\gamma_v qvB\sin\theta}{m}}$$

Emitted power in rest and observer's frame

iii) Emitted power. Use **Larmor's formula** to get the emitted power in the particle rest frame

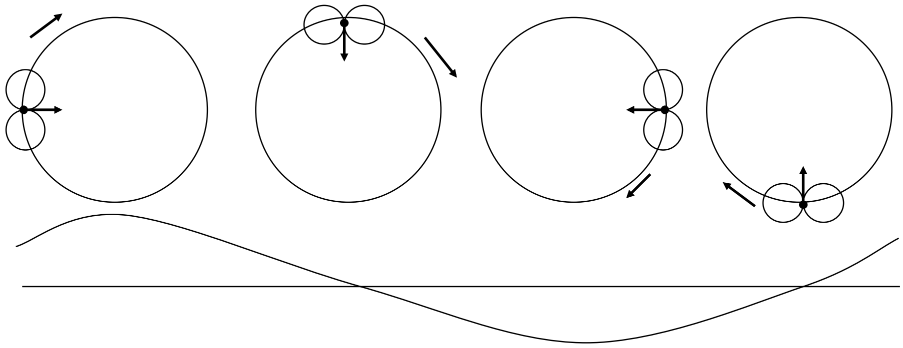
$$\frac{dW'}{dt'} = \frac{e^2}{6\pi\epsilon_0 c^3}|a'|^2 = \frac{e^2}{6\pi\epsilon_0 c^3}\left(\frac{\gamma^2 q^2 v^2}{m^2}\right) B^2 \sin^2\theta$$

Total emitted power in observer's frame is the same as in charge rest frame (*dW/dt* Lorentz invariant):

$$\frac{dW}{dt} = \frac{dW'}{dt'} = \frac{q^2}{6\pi\epsilon_0 c^3}\left(\frac{\gamma^2 q^2 v^2}{m^2}\right) B^2 \sin^2\theta$$

Gyroradiation (non-relativistic)

Power varies as $\sin^2(\omega_g t)$ so electric field varies as $\sin(\omega_g t)$. All radiation emitted at gyrofrequency of the charged particle.



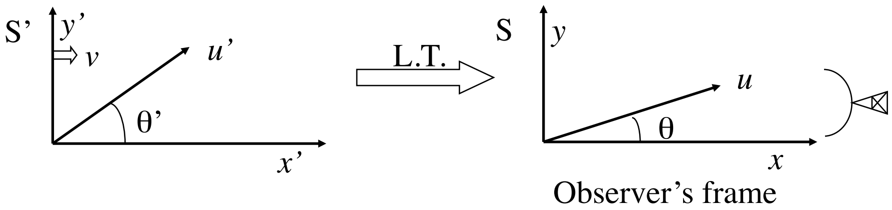
$$\omega_g = \frac{qB}{m}$$

(NB independent of velocity)

3145 Topics in Theoretical Physics - Radiation processes - Dr J Hatchell

Relativistic aberration

Results:



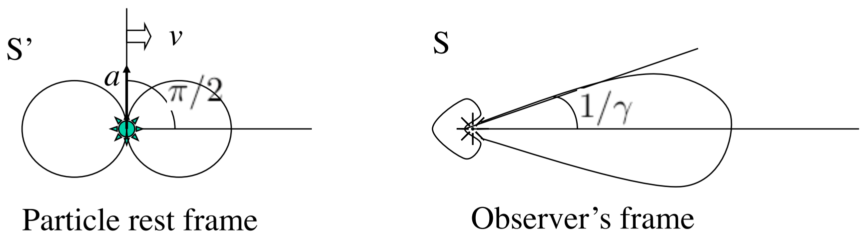
Aberration formula
$$\tan\theta = \frac{u'\sin\theta'}{\gamma(u'\cos\theta' + v)}$$

For photons $u'=c$
$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + v/c)}$$

also
$$\cos\theta = \frac{\cos\theta' + \beta}{(1 + \beta\cos\theta')}$$

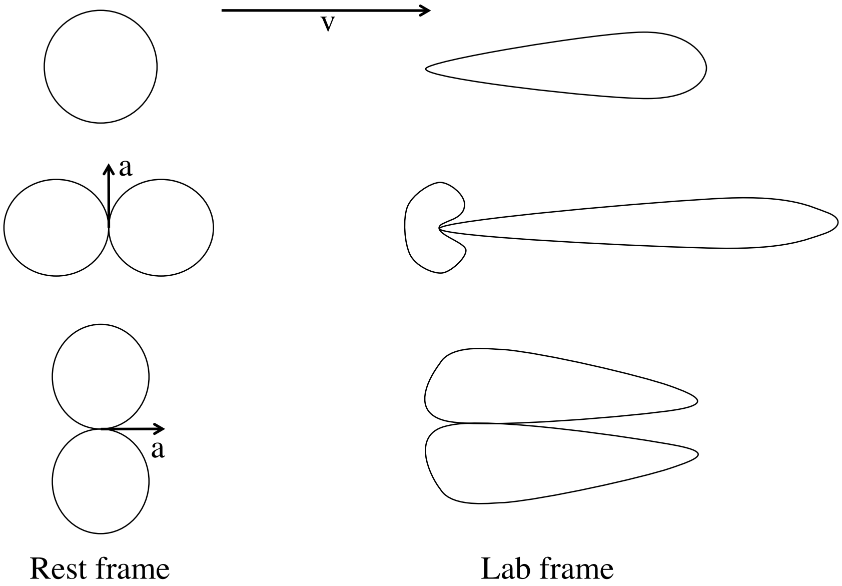
Relativistic beaming

Aim: How does the viewed power pattern of emission from an accelerated particle change when the particle moves relativistically with respect to an observer?
Quick estimate: put θ' = π/2 in aberration formula.



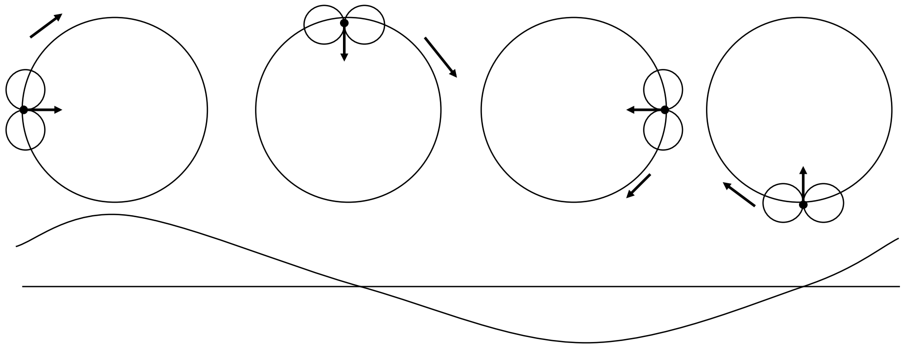
$$\theta \simeq 1/\gamma$$

2b (ii) Relativistic beaming



Gyroradiation (non-relativistic)

Power varies as $\sin^2(\omega_g t)$ so electric field varies as $\sin(\omega_g t)$. All radiation emitted at gyrofrequency of the charged particle.



$$\omega_g = \frac{qB}{m}$$

(NB independent of velocity)

Cyclotron (mildly relativistic)

Aim: Cyclotron frequency spectrum

Method: Consider increasing narrowing of pulse due to beaming (qualitatively).

Result (without derivation): Frequency spectrum consists of a series of harmonics

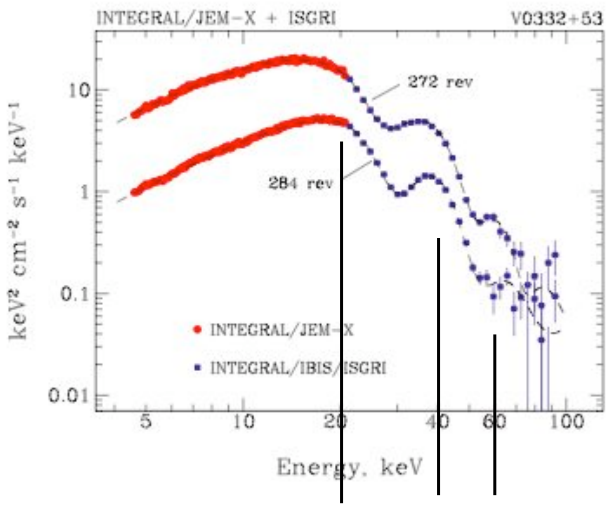
$$\omega_l = \frac{l\omega_g}{\gamma(1 - (v_{\|}/c)\cos\theta)}$$

θ is angle of B to LOS

Doppler factor for motion parallel to B

simplifying to $\omega_l = \frac{l\omega_g}{\gamma}$ for motion perpendicular to B

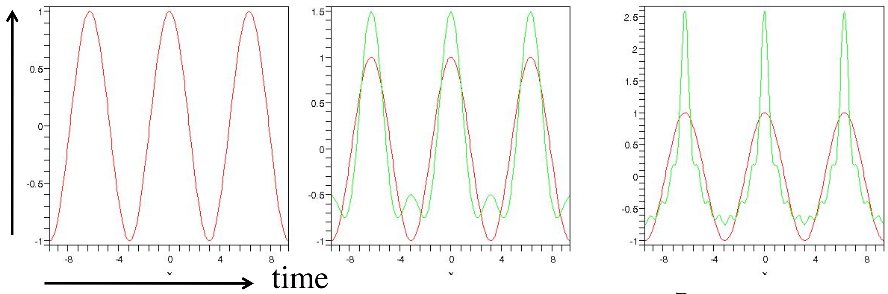
X-ray pulsar cyclotron harmonics



X-ray pulsar V0332+53

Image: Tsygankov et al. 2006 MNRAS 371, 19

Cyclotron harmonics: demonstration



$\cos(\omega_g t)$	$\cos(\omega_g t) + a_1 \cos(2\omega_g t)$	$\sum_{l=1}^{7} a_l \cos(l\omega_g t)$
Gyrofrequency only	Gyrofrequency + first harmonic	Gyrofrequency + first six harmonics
	l=1,2	l=1,2,3,4,5,6

3145 Topics in Theoretical Physics - Radiation Processes - Dr J Hatchell

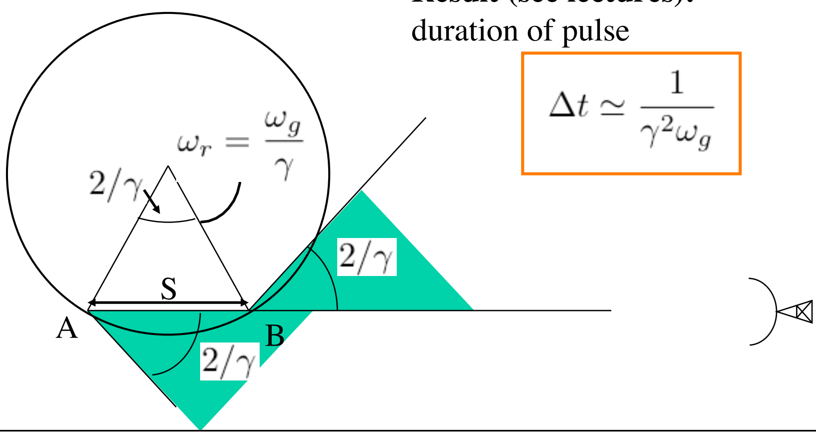
Synchrotron beaming (γ >> 1)

Aim: Understand (qualitatively) pulse length of *received* power and therefore frequency of received radiation

Method: Combine relativistic beaming (Lecture I) with relativistic motion around circle.

Result (see lectures):
duration of pulse

$$\Delta t \simeq \frac{1}{\gamma^2 \omega_g}$$

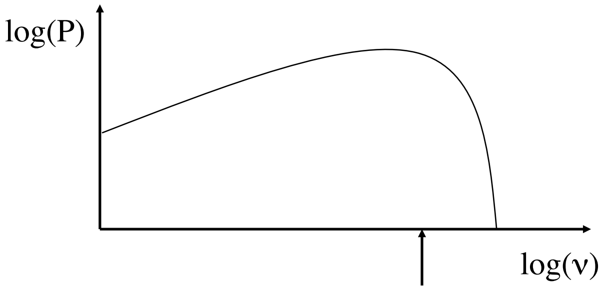


3145 Topics in Theoretical Physics - Radiation Processes - Dr J Hatchell

Synchrotron spectrum for single charge

Maximum frequency component

$$\Delta t \simeq \frac{1}{\gamma^2 \omega_g} \quad \Rightarrow \quad \nu_{\max} \sim \gamma^2 \nu_g$$

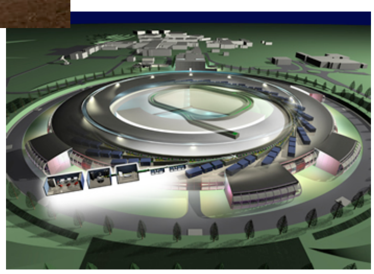


$$\nu_c = \frac{3}{2}\left(\frac{c}{v}\right)\gamma^2 \nu_g \sin\theta$$

Exact result

Diamond Synchrotron

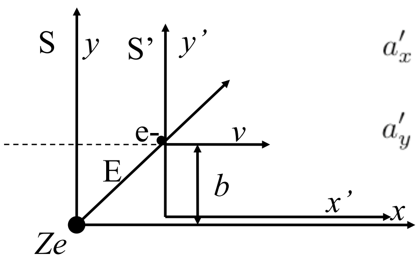
High power X-ray / UV synchrotron radiation used experimentally to probe properties of materials.



Images: www.diamond.ac.uk

i. Acceleration of electron and total power

Bremsstrahlung = **free-free radiation** = radiation by an unbound charged particle due to acceleration by another charged particle



$$a'_x = \frac{eE'_x}{m_e} = \frac{Ze^2}{4\pi\epsilon_0 m_e}\frac{\gamma v t'}{[(\gamma v t')^2 + b^2]^{3/2}}$$

$$a'_y = \frac{eE'_y}{m_e} = \frac{Ze^2}{4\pi\epsilon_0 m_e}\frac{\gamma b}{[(\gamma v t')^2 + b^2]^{3/2}}$$

Relativistic accelerations

$$P' = \frac{dW'}{dt'} = \frac{e^2}{6\pi\epsilon_0 c^3}(|a'_x(t')|^2 + |a'_y(t')|^2)$$

$$= \frac{e^2}{6\pi\epsilon_0 c^3}\left(\frac{Ze^2}{4\pi\epsilon_0 m_e}\right)^2\left[\frac{(\gamma v t')^2}{[(\gamma v t')^2 + b^2]^3} + \frac{(\gamma b)^2}{[(\gamma v t')^2 + b^2]^3}\right]$$

Total radiated power from a single electron-ion encounter